

Abelian BF Theories and Knot Invariants

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Abstract

In the context of the Batalin–Vilkovisky formalism, a new observable for the Abelian BF theory is proposed whose vacuum expectation value is related to the Alexander–Conway polynomial. The three-dimensional case is analyzed explicitly, and it is proved to be anomaly free. Moreover, at the second order in perturbation theory, a new formula for the second coefficient of the Alexander–Conway polynomial is obtained. An account on the higher-dimensional generalizations is also given.

1 Introduction

In recent years, the study of three-dimensional *topological quantum field theories* (TQFT) has shed new light on knot invariants.

The non perturbative analysis of the Chern–Simons theory in [31] has given an intrinsically three-dimensional definition of the Jones [22] and HOM-FLY [17] polynomials, while the approach of [19] has shown a more direct connection with the $(2 + 1)$ -dimensional formulation.

Later, the perturbative expansion in the covariant gauge [21, 4] has shown that numerical knot invariants can be obtained in terms of integrals over copies of the knot times copies of \mathbb{R}^3 , and the second-order invariant has been computed explicitly.

A rigorous mathematical formulation of these integrals has been given in [10] where some subtleties (“anomalies”) that arise in this framework have also been pointed out.

Another three-dimensional TQFT is the so-called BF theory [29]. The version with a cosmological term gives results that are equivalent to those obtained in Chern–Simons theory [11, 13]. The version without cosmological term (or *pure*) admits an observable [14, 12, 23] whose *vacuum expectation value* (v.e.v.) is related to the Alexander–Conway polynomial [1].

This is a “classical” knot invariant that can be defined in any odd dimension; yet it is quite difficult to find a good generalization of the three-dimensional observable of the pure BF theory in higher dimensions (s. [16] for an attempt).

However, pure BF theory is essentially an Abelian theory, as one can see by rescaling $A \rightarrow \epsilon A$ and $B \rightarrow B/\epsilon$ since, in the limit $\epsilon \rightarrow 0$, the non-Abelian perturbation $B \wedge A \wedge A$ gets killed. Therefore, we expected to get the same invariants studying the simpler Abelian version of BF theory.

In this framework, we have got to define a new non-trivial observable which, though rather involved, has a natural generalization in any odd dimension.

In the three-dimensional case, we show that this observable is not “anomalous” both in the usual field-theoretical and in the Bott and Taubes’s [10] meaning; i.e., we prove both that a quantum observable corresponding to the classical one exists and that the topological nature of its v.e.v. is not spoiled by the collapse of more than two points together (*hidden faces*).

Moreover, we show that this v.e.v. yields, at the second order in per-

turbation theory, a new integral expression for the second coefficient of the Alexander–Conway polynomial (our conjecture is that the whole v.e.v. is the inverse of the Alexander–Conway polynomial).

More generally, our approach suggests a new way of defining knot invariants (or even higher-degree forms on the space of imbeddings) as integrals over copies of the knot and a cobounding surface that turn out not to depend on the choice of the surface.

Eventually, we recall that the Abelian BF theory has a physical application [18] as a tool for studying the bosonization of many-body systems. It would be interesting to see if our observable has a physical interpretation as well.

1.1 Plan of the paper

Explicitly, in the three-dimensional case, we want to consider the classical action

$$S_{\text{cl}} = \int_{\mathbb{R}^3} B \wedge dA,$$

where A and B are one-forms. This theory is invariant under adding an exact form to either A or B . A classical observable for this theory is then given by

$$\gamma_{\text{cl}}^{(\Sigma_K, K, x_0)} = \int_{\Sigma_K} B \wedge A + \frac{1}{2} \oint_{x < y \in K} [A(x) B(y) - B(x) A(y)],$$

where $K = \partial\Sigma_K$ is a knot.

This observable can be shown to be *on-shell* invariant under the symmetries of the action, i.e., to be invariant up to terms containing the equations of motion (which state that A and B are closed forms).

Observables that are invariant only on shell can be considered in the *Batalin–Vilkovisky* (BV) formalism [6], which we briefly review in Sec. 2.

The application of the BV formalism to the three-dimensional Abelian BF theory is given in Sec. 3. (In App. A, we discuss the generalization to the case where there exist harmonic zero modes.)

In Sec. 4, we discuss the BV formulation of our observable and prove that its exponential is not anomalous, i.e., it is possible to make it an observable by adding higher-order corrections in the Planck constant. (In App. B, we find the lowest-order correction explicitly.)

In Sec. 5, we describe the Feynman diagrams of the theory, discuss the generalization to our case of the “regularization” framework developed in [10], and obtain the simplest—but non-trivial—numerical knot invariant that our theory produces, viz., the second coefficient of the Alexander–Conway polynomial. (In App. C, we prove that this v.e.v. is not anomalous in the sense of [10], and generalize this result to any order in perturbation theory.)

Finally, in Sec. 6, we describe the steps to be taken to define the higher-dimensional generalizations of our theory.

2 The BV formalism

The BV formalism [6] is a generalization of the BRST formalism [7] which is applicable to theories whose symmetry closes only on shell. Moreover, even in the case of off-shell closed symmetries, the BV formalism allows dealing with observables that are invariant only on shell, as the ones we are interested in.

In this section we give only a brief introduction. We refer to Ref. [3], whose notations we follow, for a thorough exposition of the BV formalism, as well as for a clear-cut discussion of the renormalization issue (which we have not to deal with in the present paper since the theory we consider is topological as well as Gaussian).

2.1 Preliminaries

We denote by Φ^i the *fields* one needs in a theory (i.e., the physical fields, the ghosts, the antighosts, the Lagrange multipliers and, if necessary, the ghosts for ghosts and so on); the space of fields is called the *configuration space*. We denote by $\epsilon(\Phi^i)$, or simply by ϵ_i , the ghost number of the field Φ^i . By simplicity, we consider only the case of a theory whose physical fields are bosonic, so the Grassmann parity of Φ^i is given by $(-1)^{\epsilon_i}$.

In the BV formalism, along with every field Φ^i one introduces an *antifield* Φ_i^\dagger with the same characteristics of its partner but the ghost number, which instead is given by

$$\epsilon(\Phi_i^\dagger) = -\epsilon_i - 1; \tag{1}$$

this also implies that the Grassmann parity is reversed. The space of fields

and antifields is called the *phase space* [28]. In the next sections we will also use the antifields Φ_i^* satisfying

$$\Phi_i^\dagger := *\Phi_i^*, \quad (2)$$

where $*$ is the Hodge operator.

Over the phase space one introduces a supersymplectic structure [28] which allows the definition of the BV *antibracket*

$$(X, Y) := X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}, \frac{\overrightarrow{\delta}}{\delta\Phi_i^\dagger} \right\rangle Y - X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi_i^\dagger}, \frac{\overrightarrow{\delta}}{\delta\Phi^i} \right\rangle Y \quad (3)$$

and the BV *Laplacian*

$$\Delta X := \sum_i (-1)^{\epsilon_i+1} X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}, \frac{\overrightarrow{\delta}}{\delta\Phi_i^\dagger} \right\rangle. \quad (4)$$

Here X and Y are functionals over the phase space and $\langle \cdot, \cdot \rangle$ denotes the scalar product

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta \quad (5)$$

and M is the manifold over which the theory is defined.

In Ref. [32], the BV phase space is interpreted as the tangent space $T\mathcal{M}$ over the space of fields \mathcal{M} . In the finite-dimensional case, this is locally isomorphic to the cotangent space $T^*\mathcal{M}$ by using a volume form on \mathcal{M} . Then the Laplacian Δ on $T\mathcal{M}$ is in correspondence with the exterior derivative on $T^*\mathcal{M}$. However, the product of two functionals on $T\mathcal{M}$ does not correspond to the wedge product on $T^*\mathcal{M}$, and the antibracket measures this failure since

$$(X, Y) = (-1)^{\epsilon(Y)} [\Delta(XY) - X \Delta Y - (-1)^{\epsilon(Y)} \Delta X Y] \quad (6)$$

(notice that the Laplacian as defined in (4) acts from the right).

Of course, in the infinite-dimensional case (in which we are interested), this description is only formal. In particular, the Laplacian depends on the regularization we use to define the functional integral.

2.2 BV cohomologies

One can define some cohomologies on the phase space, with respect to the gradation provided by the ghost number. Each cohomology is defined by a coboundary operator, i.e., a nilpotent operator of ghost number one.

The simplest coboundary operator is the Laplacian itself. The second interesting coboundary operator is

$$\Omega X := (X, \Sigma) - i\hbar \Delta X, \quad (7)$$

where Σ , the *quantum action*, is a bosonic functional that has to satisfy the *quantum master equation*

$$(\Sigma, \Sigma) - 2i\hbar \Delta \Sigma = 0 \quad (8)$$

for Ω to be nilpotent. Notice that (8) is equivalent to asking the Gibbs weight $\exp(i\Sigma/\hbar)$ to be Δ -closed.

The third coboundary operator we consider is

$$\sigma X := (X, S) \quad (9)$$

where S , the *action*, is a bosonic functional that has to satisfy the *master equation*

$$(S, S) = 0 \quad (10)$$

for σ to be nilpotent.

A particular case, which we encounter in this paper, is provided by a Δ -closed action S . In this case

$$\Delta \sigma + \sigma \Delta = 0, \quad (11)$$

so Δ and σ define a double complex. Moreover, by (8), S is also a quantum action for any \hbar and, as such, it defines an Ω -cohomology.

The restriction of the operator σ to the configuration space defines a new operator,

$$sX := (\sigma X)|_{\Phi^\dagger=0}, \quad (12)$$

which can be shown to be nilpotent *on shell*, i.e., modulo the solutions of

$$S_{\text{cl}} \frac{\overleftarrow{\delta}}{\delta \Phi^i} = 0, \quad (13)$$

where

$$S_{\text{cl}} = S|_{\Phi^\dagger=0} \quad (14)$$

is the *classical action*. Thus, s defines a cohomology on the configuration space on shell. If s is nilpotent also off shell, one says that the symmetry closes. In this case the BRST approach is available and s is actually the BRST operator. Notice, however, that by (12) s is defined to act from the right; the usual BRST operator s_l is the corresponding operator acting from the left, and one has

$$s_l X = (-1)^{\epsilon(X)} s X. \quad (15)$$

2.3 BV quantization

The interest in the Ω -cohomology relies on the fact that one can formally show that the class of observables whose v.e.v.'s are gauge-fixing independent is given by the Ω -closed bosonic functionals modulo Ω -exact terms. More precisely, one introduces the *partition function*

$$Z_\Psi := \int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar}\Sigma}, \quad (16)$$

where the *Lagrangian submanifold* \mathcal{L}_Ψ is defined by the equations

$$\Phi_i^\dagger = \frac{\overrightarrow{\delta}}{\delta\Phi^i} \Psi(\Phi), \quad (17)$$

and Ψ , the *gauge-fixing fermion*, is a functional on the configuration space that has ghost number -1 .

In Ref. [32], prescription (17) is shown to amount to selecting the top form in the functional on $T^*\mathcal{M}$ that, under the isomorphism between $T\mathcal{M}$ and $T^*\mathcal{M}$, corresponds to the Gibbs weight $\exp(i\Sigma/\hbar)$.

The v.e.v. of a functional X over the phase space is then defined as

$$\langle X \rangle_\Psi = \frac{1}{Z_\Psi} \int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar}\Sigma} X. \quad (18)$$

By using the formal properties of the functional integration, one has then the following [2, 28]

Statement 1 *If Σ satisfies the quantum master equation (8), then*

1. the partition function Z_Ψ and the expectation values of Ω -closed functionals do not change under infinitesimal variations of the gauge-fixing fermion Ψ , and
2. the expectation value of an Ω -exact functional vanishes.

In the finite-dimensional case, Statement 1 becomes a rigorous theorem.

One can also show that the definitions (16) and (18) correspond to the usual ones in the BRST formalism whenever applicable.

The σ -cohomology is useful since it is given by the Ω -cohomology in the limit $\hbar \rightarrow 0$ and is much easier to study. The idea is to solve the quantum master equation and to study the Ω -cohomology by an expansion in powers of \hbar . Notice, however, that from an action satisfying (10) is not always possible to obtain a quantum action satisfying (8) that, in the limit $\hbar \rightarrow 0$, yields the starting action; if this does not happen, one calls the theory *anomalous*. Moreover, even if the theory is not anomalous, a σ -closed functional of ghost number zero does not always produce an Ω -closed functional that, in the limit $\hbar \rightarrow 0$, yields the starting one. A sufficient condition for both to happen is that the one-ghost-number σ -cohomology be trivial.

As explained before, the s -cohomology is the restriction of the σ -cohomology to the configuration space on shell. Since it is easier to study than the σ -cohomology, one can study the latter by an expansion in antifields. Under some mild assumptions [30], one can prove that the extension from a classical action S_{cl} to an action S satisfying (10) and (14) exists and is unique modulo *canonical transformations* (i.e., transformations on the phase space that preserve the supersymplectic structure).

3 The three-dimensional Abelian BF theory

In this section we apply the BV formalism to the theory defined by the classical action

$$S_{\text{cl}}^\omega = \int_M B \wedge d_\omega A, \quad (19)$$

where

- M is a three-manifold;
- A and B are fields taking values in $\Omega^1(M)$;

- $d_\omega = d + i\omega$, and
- ω is an external d -closed source in $\Omega^1(M)$ (thus, $d_\omega^2 = 0$).

In the particular case $\omega = 0$, we will simply write S_{cl} and speak of the *pure* theory. We can also split S_{cl}^ω as

$$S_{\text{cl}}^\omega = S_{\text{cl}} - i\gamma_{\text{cl}}^\omega, \quad (20)$$

with

$$\gamma_{\text{cl}}^\omega = - \int_M B \wedge \omega \wedge A = \int_M \omega \wedge B \wedge A, \quad (21)$$

and see $\gamma_{\text{cl}}^\omega$ as a perturbation of the pure classical action.

If $H^1(M, d_\omega)$ is trivial (for the general case s. App. A), the symmetries of this theory are simply given by

$$\begin{aligned} s^\omega A &= d_\omega c, & s^\omega c &= 0, \\ s^\omega B &= \bar{d}_\omega \psi, & s^\omega \psi &= 0, \end{aligned} \quad (22)$$

where $\bar{d}_\omega = d - i\omega$, and c and ψ are the ghosts, which take values in $\Omega^0(M)$ and have ghost number one. Notice that $\gamma_{\text{cl}}^\omega$ is on-shell invariant under the symmetry (22) of the pure theory.

3.1 The BV action

The BV action corresponding to (19) is given by

$$S^\omega = \int_M B \wedge d_\omega A + A^* \wedge d_\omega c + B^* \wedge \bar{d}_\omega \psi + \bar{c}^* h_c + \bar{\psi}^* h_\psi, \quad (23)$$

where \bar{c} and $\bar{\psi}$ are the antighosts, and h_c and h_ψ are the Lagrange multipliers. The antighosts and the Lagrange multipliers take values in $\Omega^0(M)$; the former have ghost number minus one, the latter have ghost number zero. The additional terms in the antighosts and Lagrange multipliers are necessary to gauge fix the theory. Notice that we have used here the antifields $*$ instead of the antifields \dagger , s. (2).

If the Laplacian $d_\omega^* d_\omega + d_\omega d_\omega^*$ has zero modes, additional terms are required; for simplicity we suppose that there are no zero modes, i.e., we suppose that the cohomology $H^*(M, d_\omega)$ is trivial (s. App. A for the case when $H^1(M, d_\omega)$ is not trivial).

It is not difficult to see that S^ω satisfies the master equation (10) for any closed one-form ω . Notice moreover that $\Delta S = 0$, so S also satisfies the quantum master equation (8). Thus, we can quantize the theory with Gibbs weight $\exp(iS^\omega/\hbar)$ for any \hbar . In the following we will set $\hbar = 1$. Notice moreover that the action S^ω does not require the choice of a metric on M , so we expect its partition function to be a topological invariant of M .

The σ^ω operator (9) acts on the fields and antifields as follows

$$\begin{aligned}\sigma^\omega \psi^* &= -d_\omega B^*, & \sigma^\omega B^* &= -d_\omega A, & \sigma^\omega A &= d_\omega c, & \sigma^\omega c &= 0, \\ \sigma^\omega c^* &= -\bar{d}_\omega A^*, & \sigma^\omega A^* &= -\bar{d}_\omega B, & \sigma^\omega B &= \bar{d}_\omega \psi, & \sigma^\omega \psi &= 0,\end{aligned}\tag{24}$$

$$\begin{aligned}\sigma^\omega h_c^* &= -\bar{c}^*, & \sigma^\omega \bar{c}^* &= 0, & \sigma^\omega \bar{c} &= h_c, & \sigma^\omega h_c &= 0, \\ \sigma^\omega h_\psi^* &= -\bar{\psi}^*, & \sigma^\omega \bar{\psi}^* &= 0, & \sigma^\omega \bar{\psi} &= h_\psi, & \sigma^\omega h_\psi &= 0.\end{aligned}\tag{25}$$

It is very useful to consider the following linear combinations

$$\begin{array}{ccccccc} & (3, -2) & & (2, -1) & & (1, 0) & & (0, 1) \\ \mathcal{A} & = & -\psi^* & + & B^* & + & A & + & c, \\ \mathcal{B} & = & -c^* & + & A^* & + & B & + & \psi;\end{array}\tag{26}$$

where by (i, j) we denote an i -form of ghost number j . Notice that \mathcal{A} and \mathcal{B} have an overall (i.e., form plus ghost) degree equal to one. By (26), we can rewrite the action as

$$S^\omega = \int_M \mathcal{B} \wedge d_\omega \mathcal{A} + \bar{c}^* h_c + \bar{\psi}^* h_\psi.\tag{27}$$

Moreover, we can rewrite (24) as

$$\begin{aligned}\sigma_l^\omega \mathcal{A} &= d_\omega \mathcal{A}, \\ \sigma_l^\omega \mathcal{B} &= \bar{d}_\omega \mathcal{B},\end{aligned}\tag{28}$$

where σ_l^ω is the operator corresponding to σ^ω but acting from the left (as the exterior derivative); notice that

$$\sigma_l^\omega X = (-1)^{\epsilon(X)} \sigma^\omega X.\tag{29}$$

Following (20), we can split S^ω as

$$S^\omega = S - i\gamma^\omega,\tag{30}$$

where

$$\gamma^\omega = - \int_M \mathcal{B} \wedge \omega \wedge \mathcal{A} = \int_M \omega \wedge \tilde{\mathcal{B}} \wedge \mathcal{A}, \quad (31)$$

where the operator $\tilde{}$ acts by changing sign to odd-ghost-number terms. The splitting (30) is very convenient since not only do both S^ω and S satisfy the quantum master equation (8), but we also have

$$(\gamma^\omega, S) = 0, \quad (32)$$

$$(\gamma^\omega, \gamma^\omega) = 0, \quad (33)$$

$$\Delta \gamma^\omega = 0 \quad (34)$$

[notice that, because of (143), (32) does not hold if $H^1(M, d)$ contains non-trivial elements besides ω].

We can also split the Gibbs weight $\exp(iS^\omega)$ into the Gibbs weight $\exp(iS)$ times the observable

$$\Gamma[\omega] = \exp \gamma^\omega, \quad (35)$$

which we can prove to be Ω -closed as a consequence of (32), (33) and (34). Notice that, if ω is not trivial, the action S as to be modified as in App. A [with $b_1 = 1$ and $\varphi_1 = \overline{\varphi}_1 = v \omega' / \langle \omega', \omega' \rangle$, where $\omega' = \omega + d\alpha$ and $d^* \omega' = 0$].

By using these notations, it is easy to prove that the theory really depends only on the cohomology class of ω . In fact, if we substitute ω with $\omega + df$, the action S^ω gets an extra contribution

$$S^\omega \longrightarrow S^\omega + \Omega^\omega T^f \quad (36)$$

with

$$T^f = \int_M f \tilde{\mathcal{B}} \wedge \mathcal{A}. \quad (37)$$

By noticing that, $\Delta T^f = 0$, $(\Omega^\omega T^f, \Omega^\omega T^f) = 0$ and $(\Omega^\omega T^f, T^f) = 0$, we can show that

$$\exp(i\Omega^\omega T^f) = 1 + \Omega^\omega U^f, \quad (38)$$

with

$$U^f = \sum_{n=1}^{\infty} \frac{i^n}{n!} (\Omega^\omega T^f)^{n-1} T^f. \quad (39)$$

Since, by Statement 1, the v.e.v. of an Ω^ω -exact functional vanishes, we conclude the (formal) proof that the partition function of the theory depends only on the cohomology class of ω .

3.2 The quantization

To quantize the theory, we have to choose a gauge-fixing fermion. A convenient choice is

$$\Psi = \langle d_\omega \bar{c}, A \rangle + \langle \bar{d}_\omega \bar{\psi}, B \rangle. \quad (40)$$

Notice that to gauge fix the theory we need to choose a metric on M , but, by Statement 1, the partition function will not depend on it.

By (16) and (17), we have

$$Z[M, \omega] = \int [DA DB Dc D\bar{c} D\psi D\bar{\psi} Dh_c Dh_\psi] \exp(iS_{\text{g.f.}}^\omega), \quad (41)$$

where

$$S_{\text{g.f.}}^\omega = \int_M B \wedge d_\omega A + \langle d_\omega \bar{c}, d_\omega c \rangle + \langle \bar{d}_\omega \bar{\psi}, \bar{d}_\omega \psi \rangle + \langle h_c, d_\omega^* A \rangle + \langle h_\psi, \bar{d}_\omega^* B \rangle. \quad (42)$$

The partition function (41) can then be computed by using the zeta-function regularization of the determinants and yields [29]

$$Z[M, \omega] = \mathcal{T}(M, d_\omega), \quad (43)$$

where \mathcal{T} is the Ray–Singer torsion, which is a topological invariant of the manifold M and depends only on the cohomology class of the closed one-form ω . This explicit result confirms the previously discussed formal arguments. Notice that any multiple of ω is still a closed one-form; thus, we can consider $Z[M, \lambda\omega]$ as well.

Now we recall that the Ray–Singer torsion is equal to the Reidemeister torsion [15] and the Reidemeister torsion of the complement of a knot is proportional to the inverse of the Alexander–Conway polynomial of the knot itself [26]; thus, we can see the inverse of the Alexander–Conway polynomial as the partition function (43) of an Abelian BF theory. More precisely, we have the following (cfr. [5] and [27])

Theorem 1 *If $M = \mathbb{R}^3 \setminus \text{Tub}(K)$, where $\text{Tub}(K)$ is a tubular neighborhood of the knot $K \in \mathbb{R}^3$, and $\omega \in H^1(M)$ is such that*

$$\oint_{K_1} \omega = 1, \quad (44)$$

with K_1 a closed circle wrapping around K only once, then

$$\frac{Z[M, \lambda\omega]}{Z[M]} = \frac{1}{i\lambda} \frac{z(\lambda)}{\Delta(K; z(\lambda))}, \quad z(\lambda) = 2i \sin(\lambda/2), \quad (45)$$

where $\Delta(K; z)$ is the Alexander–Conway polynomial satisfying the skein relation

$$\Delta(K_+; z) - \Delta(K_-; z) = z \Delta(K_0; z)$$

and normalized to one on the unknot.

4 Observables for the pure theory

From now on, we will consider only the pure theory defined by the action S in (23) with $\omega = 0$. We will look for observables (i.e., Ω -closed zero-ghost-number functionals modulo Ω -exact terms) that are metric independent. By Statement 1, their v.e.v.’s will give topological invariants (up to framing) since the action is metric independent as well.

Our survey is not exhaustive; i.e., there could exist other more involved, metric-independent observables that could lead to other topological invariants.

4.1 Loop observables

The simplest observables one can build are

$$\gamma_A^K = \oint_K A, \quad \gamma_B^K = \oint_K B, \quad (46)$$

where K is an exact one-cycle [if K were only closed, these functionals would not be closed under (143)]. These observables are always Ω -exact:

$$\gamma_A^K = -\Omega\beta_\Sigma^*, \quad \gamma_B^K = -\Omega\alpha_\Sigma^*, \quad (47)$$

where

$$\alpha_\Sigma^* = \int_\Sigma A^*, \quad \beta_\Sigma^* = \int_\Sigma B^*, \quad (48)$$

and Σ is a surface cobounding K . Any function of γ_A or γ_B separately will be Ω -exact, too. To get a nontrivial observable, we have to pair them; e.g., we can consider the observable

$$\tau[K_1, K_2] = \gamma_A^{K_1} \gamma_B^{K_2}. \quad (49)$$

By (47), we can show that

$$\tau[K_1, K_2] = -i\Delta(\gamma_A^{K_1} \alpha_{\Sigma_2}^*) - \Omega(\gamma_A^{K_1} \alpha_{\Sigma_2}^*). \quad (50)$$

Since

$$\Delta(\gamma_A^{K_1} \alpha_{\Sigma_2}^*) = - \int_{\Sigma_2} \omega_{K_1} = -\#(K_1, \Sigma_2) = -\text{lk}(K_1, K_2) \quad (51)$$

(where ω_{K_1} is the Poincaré dual of K_1 , $\#$ denotes the intersection number and lk the linking number), we have

$$\langle \tau[K_1, K_2] \rangle = i \text{lk}(K_1, K_2). \quad (52)$$

An explicit computation of the l.h.s. with the gauge-fixing (40) actually gives Gauss's formula.

4.2 Surface observables

As we have seen, the loop observables are rather trivial. A more interesting observable can be built if $\dim H^1(M) = \dim H_2(M, \partial M) = 1$; viz., define

$$\gamma^\Sigma = \int_\Sigma \tilde{\mathcal{B}} \wedge \mathcal{A} = \int_\Sigma (B \wedge A + B^* \psi + c A^*), \quad (53)$$

with $\Sigma \in H_2(M, \partial M)$. This observable is essentially the same as in (31) with Σ the Poincaré dual of ω , so we know that its exponential

$$\Gamma[\Sigma, \lambda] = \exp(\lambda \gamma^\Sigma), \quad (54)$$

is an observable as well which, up to Ω -exact terms, depends only on the homology class of Σ . Moreover, the splitting (30), shows us that

$$\langle \Gamma[\Sigma, \lambda] \rangle_M = \langle \exp(\lambda \gamma^\omega) \rangle_M = \frac{Z[M, \lambda \omega]}{Z[M]}. \quad (55)$$

In particular, this holds when M is as in the hypotheses of Thm. 1. Thus, from the r.h.s. of (45) we can read the v.e.v. of $\Gamma[\Sigma, \lambda]$. Notice that condition (44) on ω requires its Poincaré dual Σ to satisfy $\#(\Sigma, K_1) = 1$. Since any surface Σ_K *spanning* the knot K (i.e., any oriented surface Σ_K imbedded in \mathbb{R}^3 such that K is identical with the boundary of Σ_K , and the orientation on Σ_K induces the given orientation on K) satisfies this property, we have the following

Theorem 2 *If $M = \mathbb{R}^3 \setminus \text{Tub}(K)$ and $\Sigma = \Sigma_K \cap M \in H_2(M, \partial M)$, then*

$$\langle \Gamma[\Sigma, \lambda] \rangle_M = \frac{1}{i\lambda} \frac{z(\lambda)}{\Delta(K; z(\lambda))}, \quad z(\lambda) = 2i \sin(\lambda/2). \quad (56)$$

Notice that a spanning surface Σ_K always exists; e.g., we can take the Seifert surface.

An expansion in powers of λ of the l.h.s. of (56) would give a representation of the coefficients of the inverse of the Alexander–Conway polynomial as Feynman diagrams involving only bivalent vertices on Σ . However, the problem of finding the propagators in a manifold like the one described above is very difficult. In the next subsection, we will see how to recast the problem as the computation of a v.e.v. in \mathbb{R}^3 .

4.3 Surface-plus-knot observables

From now on we work in \mathbb{R}^3 and consider a knot K together with a spanning surface Σ_K . Thm. 2 suggests to consider the v.e.v. of the exponential of γ^{Σ_K} . However, since Σ_K has a boundary, γ^{Σ_K} is not σ -closed anymore; actually,

$$\sigma \gamma^{\Sigma_K} = \oint_K (\psi A - Bc). \quad (57)$$

Therefore, we have to find another functional depending on K (so that it vanishes when Σ_K is closed) such that its σ -variation cancels (57). We first consider

$$\gamma^{(K, x_0)} = \frac{1}{2} \int_{x < y \in K} [A(x) B(y) - B(x) A(y)], \quad (58)$$

where x_0 is a base point on K . Notice that $K \setminus x_0$ is diffeomorphic to \mathbb{R} , so its configuration spaces $C_n(K \setminus x_0)$ are diffeomorphic to the configuration spaces $C_n(\mathbb{R})$ described in [20] (s. also subsection 5.1.3 on page 22). On these spaces it is possible to introduce the *tautological forms*

$$\eta_{ij} = \phi_{ij}^* \omega, \quad i, j = 1, \dots, n, \quad i \neq j, \quad (59)$$

where ϕ_{ij}^* denotes the pullback via the map

$$\phi_{ij}(\vec{x}) = \text{sgn}(x_i - x_j), \quad \vec{x} \in C_n(\mathbb{R}), \quad (60)$$

and $\omega = 1/2$ is the unit volume element on S^0 . With these notations, we can rewrite (58) as

$$\gamma^{(K,x_0)} = \int_{C_2(K \setminus x_0)} A_1 \wedge \eta_{12} \wedge B_2. \quad (61)$$

Now a simple computation shows that

$$\sigma \gamma^{(K,x_0)} = - \oint_K (\psi A - Bc) + \psi(x_0) \oint_K A - \oint_K B c(x_0); \quad (62)$$

so the first term cancels (62). We have then to find another functional (vanishing when Σ_K has no boundary) whose variation cancels the second and third terms. It is not difficult to see that

$$\gamma^{(\Sigma_K, x_0)} = \psi(x_0) \int_{\Sigma_K} B^* + \int_{\Sigma_K} A^* c(x_0) \quad (63)$$

does the job. Thus, we can define the following σ -closed (actually, Ω -closed) functional

$$\gamma^{(\Sigma_K, K, x_0)} = \gamma^{\Sigma_K} + \gamma^{(K, x_0)} + \gamma^{(\Sigma_K, x_0)}. \quad (64)$$

In the case of links—which we will not consider anymore in the following—the observable has to be modified as

$$\gamma^{(\Sigma_K, K, \{\Sigma_{K_i}, x_{0i}\})} = \gamma^{\Sigma_K} + \sum_i \left[\gamma^{(K_i, x_{0i})} + \gamma^{(\Sigma_{K_i}, x_{0i})} \right], \quad (65)$$

where Σ_K is a spanning surface for the link K while each Σ_{K_i} is a spanning surface only for the component K_i , whose base point is denoted by x_{0i} .

Then, recalling (54), we want to consider the exponential of $\gamma^{(\Sigma_K, K, x_0)}$,

$$\mathcal{O}_0[K, \lambda] = \exp(\lambda \gamma^{(\Sigma_K, K, x_0)}), \quad (66)$$

which is σ -closed and hence a candidate to be an observable. Actually,

$$\Delta \mathcal{O}_0[K, \lambda] = \frac{\lambda^2}{2} \mathcal{O}_0[K, \lambda] \left(\gamma^{(\Sigma_K, K, x_0)}, \gamma^{(\Sigma_K, K, x_0)} \right) \quad (67)$$

vanishes if we are working in *standard framing*, i.e., if

$$\text{slk}(K) = \int_{\Sigma_K} \omega_K = 0, \quad (68)$$

where ω_K is the Poincaré dual of K and slk denotes the self-linking number [whose definition via (68) relies on a choice of regularization]. With this hypothesis, we expect the v.e.v. of \mathcal{O}_0 not to depend on the gauge fixing and, as a consequence, to be metric independent.

By essentially the same proof that led to the invariance (modulo Ω -exact terms) of $\Gamma[\omega]$, s. (35), under $\omega \rightarrow \omega + d\eta$, we can prove that \mathcal{O}_0 is invariant (modulo Ω -exact terms) under $\Sigma_K \rightarrow \Sigma_K + \partial T$ with $T \in \Omega_3(\mathbb{R}^3)$.

From (56) we expect the v.e.v. of $\mathcal{O}_0[K, \lambda]$ to be proportional to the inverse of the Alexander–Conway polynomial. The proportionality constant, which depends on λ , could be spoiled when we send $\mathbb{R}^3 \setminus \text{Tub}(K)$ to \mathbb{R}^3 ; thus, we can make only the weaker statement that

$$\frac{\langle \mathcal{O}_0[K_{\text{s.f.}}, \lambda] \rangle}{\langle \mathcal{O}_0[\bigcirc_{\text{s.f.}}, \lambda] \rangle} = \frac{1}{\Delta(K; z(\lambda))}, \quad z(\lambda) = 2i \sin(\lambda/2), \quad (69)$$

where $K_{\text{s.f.}}$ and $\bigcirc_{\text{s.f.}}$ are, respectively, a generic knot and the unknot in standard framing.

This result has to be compared with the similar formulae obtained in the context of the *non-Abelian* pure BF theory [14, 12]. It should not amaze that the Abelian and non-Abelian pure BF theories are under this respect equivalent, for, as observed in [12], the v.e.v.’s of the latter can be computed exactly in saddle-point approximation (s. also the observation in the Introduction).

We want to point out the precise meaning of (69): We are not claiming that the coefficients of the λ -expansion of $\langle \mathcal{O}_0 \rangle$ are a sum of numerical knot invariants up to factors containing the self-linking number. We are saying that these numerical knot invariants are well defined only if the knot is in standard framing; otherwise \mathcal{O}_0 is not an observable, and we are not guaranteed that its v.e.v. is a topological invariant.

This means that, to compute this v.e.v., we have to choose a particular presentation of the knot, viz., one in which the self-linking number vanishes, and that this v.e.v. should be invariant only under deformations that do not change the self-linking number. In the next section, we will discuss how to drop this cumbersome condition.

4.4 The corrected surface-plus-knot observable

For the purposes of this subsection, it is convenient to rescale

$$\mathcal{B} \longrightarrow \mathcal{B}/\lambda, \quad (70)$$

so the Gibbs weight becomes $\exp(iS/\lambda)$ and we recognize λ as the Planck constant of the theory. Under this rescaling we also have

$$\mathcal{O}_0[K, \lambda] \longrightarrow \mathcal{O}_0[K] = \exp \gamma^{(\Sigma_K, K, x_0)}; \quad (71)$$

thus, \mathcal{O}_0 is a classical observable, i.e., it is σ -closed and does not depend on the Planck constant. Now we look for a quantum generalization

$$\mathcal{O}[K, \lambda] = \sum_{n=0}^{\infty} (i\lambda)^n \mathcal{O}_n[K] \quad (72)$$

satisfying

$$\Omega \mathcal{O} = (\sigma - i\lambda \Delta) \mathcal{O} = 0, \quad (73)$$

and hence

$$\sigma \mathcal{O}_n = \Delta \mathcal{O}_{n-1}, \quad n = 1, 2, \dots \quad (74)$$

Notice that a solution \mathcal{O}_n , if it exists, is defined only up to σ -closed terms. However, if we send \mathcal{O}_n into $\mathcal{O}_n + \sigma Y_n$, the solution \mathcal{O}_{n+1} will be sent into $\mathcal{O}_{n+1} - \Delta Y_n$ because of (11). Thus, \mathcal{O} will be changed by an Ω -exact term. On the other hand, if we add a nontrivial σ -closed term to \mathcal{O}_n , then this—
together with the extra contribution \mathcal{O}_{n+1} receives by (74)—lets \mathcal{O} get an extra Ω -closed term. This means that the solution of (73), if it exists, is unique only up to Ω -closed terms, i.e., up to other observables.

It is not difficult to see, by (11), that, if (74) holds up to a fixed $n - 1$, then

$$\sigma \Delta \mathcal{O}_{n-1} = 0. \quad (75)$$

Thus, the r.h.s. of (74) is σ -closed; however, to solve (74), we want it to be σ -exact, which is not guaranteed. If this happens, then the observable \mathcal{O} exists and we say that \mathcal{O}_0 is not anomalous.

Among the possible solutions of (73), we are interested in the ones that depend only on the triple Σ_K, K, x_0 and that reduce to $\exp \gamma^{\Sigma_K}$ when Σ_K has no boundary. We call these solutions *proper*.

Now notice that the action is invariant under the change of variables $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{B}, \mathcal{A})$ while the observable $\gamma^{(\Sigma_K, K, x_0)}$ is odd under it. This means that the corrections can be chosen to have a well-defined parity. By induction, one can see that they can be written as integrals of products of $\mathcal{B} \wedge \mathcal{A}$ and $\tilde{\mathcal{B}} \wedge \mathcal{A}$ over submanifolds of products of configuration spaces of \mathbb{R}^3 . Moreover, $\Delta \mathcal{O}_n$ will have the same structure. Since $\tilde{\mathcal{B}} \wedge \mathcal{A}$ and $\mathcal{B} \wedge \mathcal{A}$ are overall two-forms (i.e., each component has form degree plus ghost number equal to two), a product of them will be an overall even form. As a consequence, the zero-ghost-number part will be an even form, while the one-ghost-number part will be an odd form. However, only the even homology spaces of the configuration spaces of \mathbb{R}^3 are nontrivial. Therefore, since \mathcal{O}_n has ghost number zero, it can be a non-trivial element of the σ -cohomology, whereas $\Delta \mathcal{O}_n$, which has ghost number one and is σ -closed, must be σ -exact. Thus, we have proved the following

Theorem 3 *The classical observable $\mathcal{O}_0[K]$ is not anomalous and admits a proper extension.*

By Statement 1, we expect the v.e.v. of \mathcal{O} to be a topological invariant (up to framing) of the triple Σ_K, K, x_0 . If the argument proving the invariance (modulo Ω -exact terms) under a deformation $\Sigma_K \rightarrow \Sigma_K + \partial T$ with $T \in \Omega_3(\mathbb{R}^3)$ goes through, we arrive at the following

Conjecture 1 *The v.e.v. of a proper solution $\mathcal{O}[K, \lambda]$ is a regular-isotopy invariant of the knot K .*

Now write this invariant as the sum of an ambient-isotopy invariant and a regular-isotopy invariant that vanishes in standard framing. If the second contribution can be written as the v.e.v. of an Ω -closed observable, then we can redefine the proper solution \mathcal{O} by subtracting this term; so we arrive to the following

Conjecture 2 *There exists a proper solution whose v.e.v. is an ambient-isotopy invariant.*

Eventually, since \mathcal{O} is a quantum generalization of \mathcal{O}_0 to which it reduces in standard framing, and the v.e.v. of \mathcal{O}_0 is expected to satisfy (69), we have our last

Conjecture 3 *The proper solution $\mathcal{O}[K, \lambda]$ of Conjecture 2 satisfies*

$$\frac{\langle \mathcal{O}[K, \lambda] \rangle_\lambda}{\langle \mathcal{O}[\bigcirc, \lambda] \rangle_\lambda} = \frac{1}{\Delta(K; z(\lambda))}, \quad z(\lambda) = 2i \sin(\lambda/2), \quad (76)$$

where \bigcirc denotes the unknot.

If only Conjecture 1 holds, we still expect (76) to hold but only if the standard framing is chosen.

4.4.1 The first correction

In App. B, we discuss how to find the first correction to \mathcal{O}_0 . In particular, we show that a proper solution is given by

$$\mathcal{O}_1 = \mathcal{O}_0 U_1, \quad (77)$$

with

$$U_1 = \Delta u_1, \quad (78)$$

and

$$u_1 = \gamma_{ABB} \int_{\Sigma_K} B^* + \gamma_{BAA} \int_{\Sigma_K} A^*, \quad (79)$$

where

$$\gamma_{ABB} = \int_{C_3(K \setminus x_0)} A_1 \eta_{12} B_2 \eta_{23} B_3 = -\frac{1}{2} \int_{x < y < z \in K} B(x) A(y) B(z), \quad (80)$$

$$\gamma_{BAA} = \int_{C_3(K \setminus x_0)} B_1 \eta_{12} A_2 \eta_{23} A_3 = -\frac{1}{2} \int_{x < y < z \in K} A(x) B(y) A(z). \quad (81)$$

Remember that \mathcal{O}_1 is defined up to a σ -closed term. Our choice—(77), (78) and (79)—is particularly convenient since it gives the correct v.e.v. of \mathcal{O} to the second order in λ . To see this, we first observe that any Wick contraction in the computation of a v.e.v. carries a factor λ . Thus, at order λ^2 , the v.e.v. of $\mathcal{O}_0 + i\lambda \mathcal{O}_1$ will contain the v.e.v.'s of $1/2(\gamma^{(\Sigma_K, K, x_0)})^2$ and of $i\lambda U_1$. Since $\Delta U_1 = 0$, we have

$$\Omega O_2 = 0, \quad (82)$$

where

$$O_2 = \left(\frac{1}{2} (\gamma^{(\Sigma_K, K, x_0)})^2 + i\lambda U_1 \right); \quad (83)$$

therefore, no other correction is needed to make this second-order term an observable.

Notice that any redefinition of U_1 obtained by adding a Δ -closed term will have the same property. By (74), this additional term must also be σ -closed, so it will be Ω -closed as well. Thus, as expected, O_2 is defined up to Ω -closed terms whose v.e.v.'s are of order λ^2 . There exist only a few of such terms, i.e., λ^2 , $\lambda\tau[K]$ and $\tau[K]^2$, where [cfr. (49)]

$$\tau[K] = \gamma_A^K \gamma_B^K. \quad (84)$$

By (52)—and remembering (70)—we see that

$$\langle O_2 + k\tau^2 + i\lambda l\tau - \lambda^2 m \rangle_\lambda = \langle O_2 \rangle_\lambda - \lambda^2 [2k(\text{slk}K)^2 + l\text{slk}K + m]; \quad (85)$$

i.e., the second order of $\langle \mathcal{O}[K, \lambda] \rangle_\lambda$ is defined up to a quadratic function of the self-linking number of K . We will see in the next section that, choosing $k = -3/16$ and $l = m = 0$, Conjectures 1, 2 and 3 hold at this order.

Remark Notice that we are allowed to add to O_2 only contributions of the form λ times an observable, for λ -independent contributions would change the classical part of the observable. Thus, τ^2 would not be an allowed contribution. However, as shown at the end of App. B, adding τ^2 to O_2 is equivalent to adding a λ -independent correction to U_1 , s. (159).

5 Computation of v.e.v.'s

In this section we will describe the perturbative expansion of $\langle \mathcal{O}_0 \rangle_\lambda$ with the gauge fixing (40). We will also explicitly compute the second order term of $\langle \mathcal{O}_0 \rangle$ and $\langle \mathcal{O} \rangle$. We will see that the latter satisfies, at this order, Conjectures 1 and 3.

5.1 Gauge fixing and propagators

We will work in the covariant gauge fixing; i.e., we will choose the gauge-fixing fermion as in (40) with $\omega = 0$. By (17) and (2), this amounts to setting

$$\begin{aligned} A^* &= *d\bar{c}, & \bar{c}^* &= *d^*A, \\ B^* &= *d\bar{\psi}, & \bar{\psi}^* &= *d^*B, \end{aligned} \quad (86)$$

while all the other antifields are set to zero. Thus, the gauge-fixed action reads [cfr. (42)]

$$S_{\text{g.f.}} = \int_{\mathbb{R}^3} B \wedge dA + \langle d\bar{c}, dc \rangle + \langle d\bar{\psi}, d\psi \rangle + \langle h_c, d^*A \rangle + \langle h_\psi, d^*B \rangle, \quad (87)$$

from which we can read the propagators (we write only the ones we are interested in)

$$\begin{aligned} \langle A_\mu(x) B_\nu(y) \rangle_\lambda &= \langle B_\mu(x) A_\nu(y) \rangle_\lambda = i\lambda \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}, \\ \langle c(x) \bar{c}(y) \rangle_\lambda &= -\langle \bar{c}(x) c(y) \rangle_\lambda = i\lambda \frac{1}{4\pi} \frac{1}{|x-y|}, \\ \langle \psi(x) \bar{\psi}(y) \rangle_\lambda &= -\langle \bar{\psi}(x) \psi(y) \rangle_\lambda = i\lambda \frac{1}{4\pi} \frac{1}{|x-y|}. \end{aligned} \quad (88)$$

Notice that the propagators are exactly the same as in Chern–Simons theory [21, 4].

5.1.1 Parity

We have already observed that, under $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{B}, \mathcal{A})$, the action is left unchanged while $\gamma^{(\Sigma_K, K, x_0)}$ changes sign. As a consequence, in the gauge-fixing defined by (86), the propagators (88) are invariant under

$$(A, B, c, \bar{c}, \psi, \bar{\psi}, h_c, h_\psi) \rightarrow (B, A, \psi, \bar{\psi}, c, \bar{c}, h_\psi, h_c). \quad (89)$$

Thus, all the terms in perturbation theory that are odd under (89) [like, e.g., the v.e.v. of $(\gamma^{(\Sigma_K, K, x_0)})^{2n+1}$] will vanish.

5.1.2 Supersymmetry

Another observation that will simplify the discussion of the perturbation theory concerns the supersymmetry of the action (87) (which is the same that holds in its non-Abelian generalization [24]); viz., we can define a fermionic vector operator Q , i.e., an operator satisfying

$$[Q_\alpha, Q_\beta]_+ = Q_\alpha Q_\beta + Q_\beta Q_\alpha = 0, \quad (90)$$

which annihilates the action:

$$QS = 0. \quad (91)$$

Actually, there exist two such operators (which, moreover, anticommute with each other); the first one acts as

$$\begin{aligned}
(QA)_{\alpha\beta} &= \epsilon_{\alpha\beta\gamma} \partial^\gamma \bar{\psi}, \\
(QB)_{\alpha\beta} &= \epsilon_{\alpha\beta\gamma} \partial^\gamma \bar{c}, \\
(Qc)_\alpha &= -A_\alpha, \\
(Q\psi)_\alpha &= -B_\alpha, \\
(Q\bar{c})_\alpha &= 0, \\
(Q\bar{\psi})_\alpha &= 0, \\
(Qh_c)_\alpha &= \partial_\alpha \bar{c}, \\
(Qh_\psi)_\alpha &= \partial_\alpha \bar{\psi}.
\end{aligned} \tag{92}$$

The second operator is obtained by exchanging (c, \bar{c}, h_c) with $(\psi, \bar{\psi}, h_\psi)$.

A consequence of this supersymmetry is that the v.e.v. of a Q -exact functional vanishes. We want to point out that this supersymmetry is peculiar of \mathbb{R}^3 , but holds (with a different Q -operator) also for other gauge fixings.

5.1.3 Regularization

The propagators (88) diverge as the two points where the fields are evaluated approach to each other. The non-regularized v.e.v.'s of our observables are integrals of these propagators over products of $C_k(K \setminus x_0)$ and Σ_K . To avoid divergences we have to give a prescription to split the points in these integrals.

Our choice will essentially follow the approach of [10] with some important modifications due to the presence of the surface Σ_K (s. also [23]). The idea is to start defining the Fulton–MacPherson [20] compactification $C_n(\mathbb{R}^3)$ of the configuration space of n points in $\overline{\mathbb{R}^3}$, where the latter is the compactification of \mathbb{R}^3 obtained by replacing the infinity with its blow up. Then, denoting by B^2 a two-dimensional surface whose boundary is diffeomorphic to S^1 , we can consider the following imbeddings of compact spaces

$$\text{pt} \hookrightarrow S^1 \hookrightarrow \overline{B^2} \hookrightarrow \overline{\mathbb{R}^3}, \tag{93}$$

where pt is a base point on the sphere S^1 which is mapped to the boundary of B^2 . This allows us to define the configuration space C_n^t of n points on the knot distinct from the base point and t points on its spanning surface. Notice that the points on the knot can be ordered, so C_n^t has $n!$ connected components.

We will denote by \tilde{C}_n^t the identity component (i.e., the component with points on S^1 ordered as $0, 1, 2, \dots, n$).

Our regularization prescription to compute the v.e.v.'s will be to replace $C_k(K \setminus x_0)^n \times (\Sigma_K)^t$ with C_{kn}^t . Moreover, we will rewrite the propagators (88) as

$$\begin{aligned} \langle A_i \wedge B_j \rangle_\lambda &= \langle B_i \wedge A_j \rangle_\lambda = i\lambda \theta_{ij}, \\ \langle c_i (*d\bar{c})_j \rangle_\lambda &= \langle (*d\bar{c})_i c_j \rangle_\lambda = -i\lambda \theta_{ij}, \\ \langle \psi_i (*d\bar{\psi})_j \rangle_\lambda &= \langle (*d\bar{\psi})_i \psi_j \rangle_\lambda = -i\lambda \theta_{ij}, \end{aligned} \quad (94)$$

where θ_{ij} is the tautological form on \mathbb{R}^3 defined as the pullback of the $SO(3)$ -invariant unit volume element on S^2 by the map

$$\phi_{ij}(\vec{x}) = \frac{x_j - x_i}{|x_j - x_i|}, \quad \vec{x} \in C_n(\mathbb{R}^3). \quad (95)$$

Notice that θ_{ij} satisfies

$$d\theta_{ij} = 0, \quad \theta_{ij}^2 = 0, \quad \theta_{ji} = -\theta_{ij}. \quad (96)$$

To compute our v.e.v.'s, we will have to integrate these two-forms as well as the tautological zero-forms η_{ij} [appearing in (61)] over some C_n^t . If we choose the identity component \tilde{C}_n^t , we can eliminate the zero-forms η . Thus, we can represent the contributions to our v.e.v.'s graphically as follows: we represent the knot as a horizontal line (which we suppose directed from left to right) with the base point on its boundary and the spanning surface as the portion of plane above it, and the two-forms θ_{ij} as arrows connecting the point i to the point j (s. fig. 1).

We can give even a better description—cfr. [10]—if we introduce the bundles

$$\begin{array}{c} \tilde{C}_n^t(S^1 \times \mathcal{P} \times \mathcal{S}) \\ \downarrow p \\ S^1 \times \mathcal{P} \times \mathcal{S} \end{array}, \quad (97)$$

where \mathcal{P} is the space of the maps $S^1 \rightarrow S^1$ homotopic to the identity and \mathcal{S} is the space of imbeddings of B^2 in \mathbb{R}^3 , and the fiber is \tilde{C}_n^t . Now we can see a generic v.e.v. as the sum of contributions which read

$$I = p_*[I], \quad (98)$$

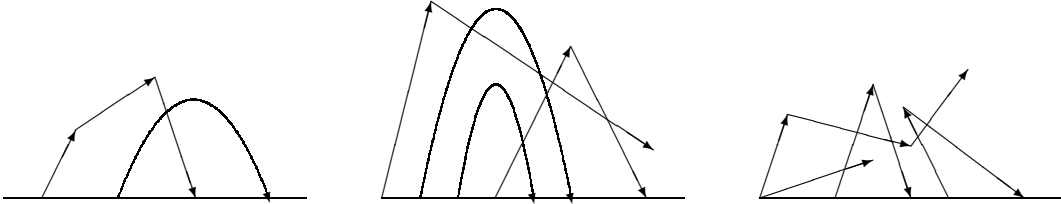


Figure 1: Some examples of diagrams with nonvanishing v.e.v.

where p_* denotes the push forward along the fiber, and $[I]$ is a form on the bundle.

A locally constant function on the base space—i.e., a function whose differential vanishes)—will be a topological invariant on \mathcal{S} , for a locally constant function is a constant on $S^1 \times \mathcal{P}$ which are connected. If, moreover, this topological invariant does not depend on cohomological deformations of the surface, it will eventually be an invariant of its boundary.

As in [10], the differential of I can be written, by Stokes's theorem, as

$$dI = p_* d[I] + p_*^\partial [I] = p_*^\partial [I], \quad (99)$$

where p_*^∂ denotes the push forward along the boundary of the fiber.

It is useful to distinguish on this boundary between *principal* and *hidden* faces. The principal faces are essentially of four types:

1. two points on the knot collapse together;
2. one point on the knot collapses to the base point;
3. two point on the surface collapse together;
4. one point on the surface collapses on the knot where either (a) there are no points, or (b) there is one point, or (c) there is the base point.

All the other components of the boundary (viz., when more points come together) are referred to as the hidden faces. Among the principal faces, we will call *simple* the ones of type 1, 2 and 3a.

The principal-face contribution δI to dI can be evaluated “graphically” just by looking at the diagrams. What is not immediate is seeing whether the push forward along the hidden faces vanishes. However, in App. C we prove a vanishing theorem for the push forward along all faces but the simple principal faces (s. Thm. 7).

5.2 The perturbative expansion of $\langle \mathcal{O}_0 \rangle$

By (86), the gauge-fixed observable $\gamma^{(\Sigma_K, K, x_0)}_{\text{g.f.}}$ (64) now reads

$$\gamma^{(\Sigma_K, K, x_0)}_{\text{g.f.}} = \gamma^{\Sigma_K}_{\text{g.f.}} + \gamma^{(K, x_0)}_{\text{g.f.}} + \gamma^{(\Sigma_K, x_0)}_{\text{g.f.}}, \quad (100)$$

with

$$\gamma^{\Sigma_K}_{\text{g.f.}} = \int_{\Sigma_K} [B \wedge A + (*d\bar{\psi})\psi + c(*d\bar{c})], \quad (101)$$

$$\gamma^{(K, x_0)}_{\text{g.f.}} = \int_{C_2(K \setminus x_0)} A_1 \eta_{12} \wedge B_2, \quad (102)$$

$$\gamma^{(\Sigma_K, x_0)}_{\text{g.f.}} = \psi(x_0) \int_{\Sigma_K} (*d\bar{\psi}) + \int_{\Sigma_K} (*d\bar{c}) c(x_0). \quad (103)$$

5.2.1 The general structure of the perturbative expansion

The first thing we notice is that all these functionals are odd under (89). Thus, only an even product of them will have a nonvanishing v.e.v. This, by the way, proves that $\langle \mathcal{O}_0 \rangle_\lambda$ is even in λ , in accordance with (69).

The second observation is that, by (92), the functional $\gamma^{\Sigma_K}_{\text{g.f.}}$ is Q -exact, viz.,

$$\gamma^{\Sigma_K}_{\text{g.f.}} = \int_{\Sigma_K} dx^\alpha \wedge dx^\beta [Q(\psi A - Bc)]_{\alpha\beta}; \quad (104)$$

thus, the v.e.v. of any of its powers vanishes. This implies that no loops appear among the v.e.v.’s we are computing. In fig. 2.a, we show one of these loops. Notice that, if we were working on a less trivial manifold, (87) would not be supersymmetric, so such loops would exist. As a matter of fact, the v.e.v. considered in (56) consists entirely of such loop diagrams.

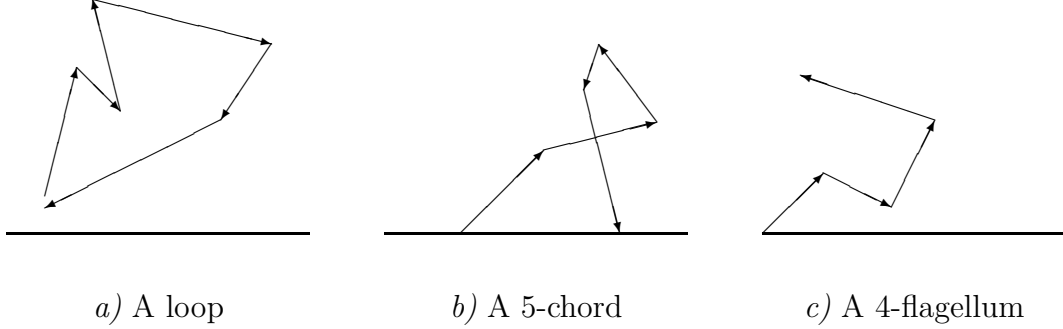


Figure 2: Elements that appear in the diagrams

The third observation is that $\gamma_{\text{g.f.}}^{(\Sigma_K, x_0)}$ is linear in the Grassmann variables $c(x_0)$ and $\psi(x_0)$. Thus, its square simply reads

$$(\gamma_{\text{g.f.}}^{(\Sigma_K, x_0)})^2 = \psi(x_0) \int_{\Sigma_K} (*d\bar{\psi}) \int_{\Sigma_K} (*d\bar{c}) c(x_0), \quad (105)$$

while all higher powers vanish.

We can now describe the features of the perturbative expansion of $\langle \mathcal{O}_0 \rangle_\lambda$. In the v.e.v.'s where no $\gamma_{\text{g.f.}}^{(\Sigma_K, x_0)}$ appears, we have to Wick contract the fields A and B on K and on Σ_K together in all possible ways discarding all the diagrams that contain a loop. Thus, we are left with chains that connect two points on K through a certain number of bivalent vertices on Σ_K . We will call an n -*chord* such a chain, where $n > 0$ is the number of links (s. fig. 2.b).

If the v.e.v. contains $\gamma_{\text{g.f.}}^{(\Sigma_K, x_0)}$, besides the chords described above, we will have a chain connecting x_0 with a point on Σ_K through a certain number of bivalent vertices on Σ_K . We will call an n -*flagellum* such a chain (s. fig. 2.c).

Eventually, we have v.e.v.'s containing $(\gamma_{\text{g.f.}}^{(\Sigma_K, x_0)})^2$. They contain some chords and two flagella.

For a v.e.v. not to vanish, the total number of links in all the chords must be even; moreover, the total number of links in the flagella must be even.

In figs. 1 and 3, some diagrams of nonvanishing v.e.v.'s are shown. At order $2n$ in λ , we will have a sum of diagrams of this kind with a total number of links equal to $2n$.

From the structure of the perturbative expansion, it is easy to see that, if $\langle \mathcal{O}_0 \rangle_\lambda$ is a topological invariant, then it is invariant under $\Sigma_K \rightarrow \Sigma_K + \partial T$

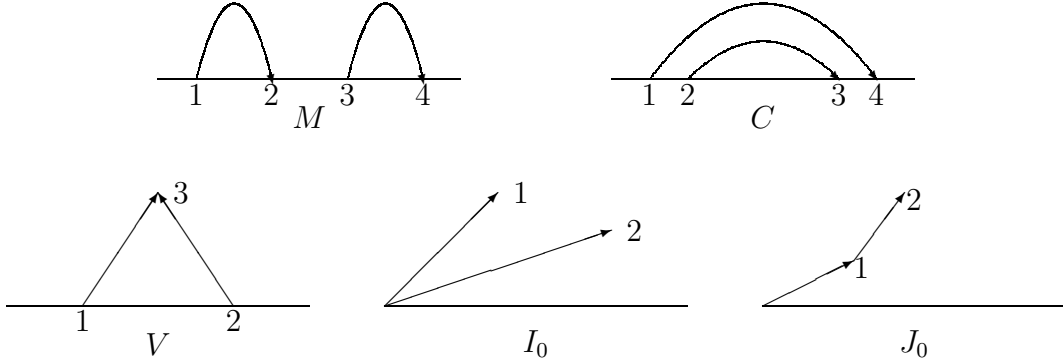


Figure 3: The second-order diagrams

with $T \in \Omega_3(\mathbb{R}^3)$. In fact, if we have a topological invariant, we can move the region where this deformation occurs to infinity. Since all the vertices on Σ_K are connected through a finite number of links to a point on K , this region at infinity does not contribute. This, of course, would not be true if loops on Σ_K were allowed.

As a final remark, we notice that, if we represent the unknot in standard framing as a planar curve and choose its spanning surface to belong to the same plane, then

$$\langle \mathcal{O}_0[\bigcirc_{\text{s.f.}}, \lambda] \rangle = 1. \quad (106)$$

5.2.2 The second order

Now we want to compute explicitly the v.e.v. of $\frac{1}{2}(\gamma^{(\Sigma_K, K, x_0)})^2$. By the supersymmetry argument, we know that the v.e.v. of $(\gamma^{\Sigma_K})^2$ vanishes. Moreover, the v.e.v. of $\gamma^{(K, x_0)}\gamma^{(\Sigma_K, x_0)}$ is equal to the product of the v.e.v.'s of $\gamma^{(K, x_0)}$ and $\gamma^{(\Sigma_K, x_0)}$, which vanish. Thus, we are left with only four contributions,

which, after some computations, can be written as

$$\begin{aligned}
\left\langle \frac{1}{2}(\gamma^{(K,x_0)})^2 \right\rangle_\lambda &= \lambda^2 (M - C), \\
\left\langle \gamma^{(K,x_0)} \gamma^{\Sigma_K} \right\rangle_\lambda &= \lambda^2 V, \\
\left\langle \frac{1}{2}(\gamma^{(\Sigma_K,x_0)})^2 \right\rangle_\lambda &= \lambda^2 I_0, \\
\left\langle \gamma^{\Sigma_K} \gamma^{(\Sigma_K,x_0)} \right\rangle_\lambda &= -2\lambda^2 J_0,
\end{aligned} \tag{107}$$

where the diagrams M , C , V , I_0 and J_0 are shown in fig. 3. Explicitly they read

$$\begin{aligned}
M &= \int_{\tilde{C}_4^0} \theta_{12} \wedge \theta_{34}, \\
C &= \int_{\tilde{C}_4^0} \theta_{14} \wedge \theta_{23}, \\
V &= \int_{\tilde{C}_2^1} \theta_{13} \wedge \theta_{23}, \\
I_0 &= \int_{\tilde{C}_0^2} \theta_{01} \wedge \theta_{02}, \\
J_0 &= \int_{\tilde{C}_0^2} \theta_{01} \wedge \theta_{12}.
\end{aligned} \tag{108}$$

Thus, the second order of $\langle \mathcal{O}_0 \rangle_\lambda$ reads

$$\left\langle \frac{1}{2}(\gamma^{(\Sigma_K, K, x_0)})^2 \right\rangle_\lambda = \lambda^2 (M - C + V + I_0 - 2J_0). \tag{109}$$

5.3 The v.e.v. of the corrected observable

As explained in subsection 4.3, we do not expect $\langle \mathcal{O}_0 \rangle_\lambda$ to be a knot invariant (at least not with a general framing), since, in general, \mathcal{O}_0 is *not* an observable. In subsection 4.4, we have seen that there is a procedure that leads to an observable \mathcal{O} starting from \mathcal{O}_0 . We have computed the first correction (77) explicitly and have shown that the corrected second-order v.e.v. is given by the v.e.v. of the observable \mathcal{O}_2 defined in (83). In this section, we will compute this v.e.v. explicitly and show that it is a knot invariant.

5.3.1 The v.e.v. of \mathcal{O}_2

To evaluate the v.e.v. of the correction $i\lambda U_1$, we notice that, by (78) and Statement 1,

$$\langle i\lambda U_1 \rangle_\lambda = \langle i\lambda \Delta u_1 \rangle_\lambda = \langle \sigma u_1 \rangle_\lambda. \tag{110}$$

By (79), we have then

$$\langle i\lambda U_1 \rangle_\lambda = \tilde{U}_1 + \tilde{U}_2, \tag{111}$$

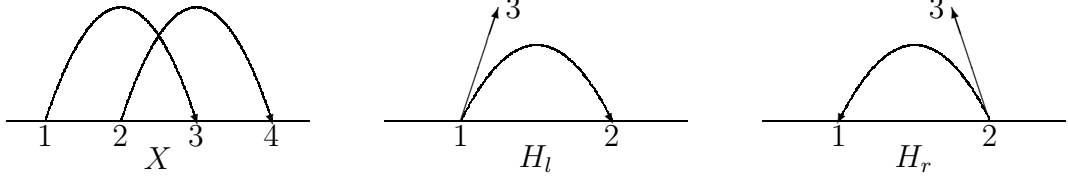


Figure 4: The second-order correction diagrams

with

$$\begin{aligned}\tilde{U}_1 &= \left\langle \gamma_{ABB} \sigma \int_{\Sigma_K} B^* + \gamma_{BAA} \sigma \int_{\Sigma_K} A^* \right\rangle_\lambda, \\ \tilde{U}_2 &= -\left\langle (\sigma \gamma_{ABB}) \int_{\Sigma_K} B^* + (\sigma \gamma_{BAA}) \int_{\Sigma_K} A^* \right\rangle_\lambda.\end{aligned}\quad (112)$$

Finally, an explicit evaluation of this v.e.v.'s yields

$$\begin{aligned}\tilde{U}_1 &= 2\lambda^2 (X - 2M - C), \\ \tilde{U}_2 &= \lambda^2 (H_l + H_r),\end{aligned}\quad (113)$$

where the new diagrams X , H_l and H_r are shown in fig. 4. Explicitly they read

$$\begin{aligned}X &= \int_{\tilde{C}_4^0} \theta_{13} \wedge \theta_{24}, \\ H_l &= \int_{\tilde{C}_2^1} \theta_{12} \wedge \theta_{13}, \\ H_r &= \int_{\tilde{C}_2^1} \theta_{21} \wedge \theta_{23}.\end{aligned}\quad (114)$$

Therefore, the correction to $\langle \mathcal{O}_0 \rangle_\lambda$ reads

$$\langle i\lambda U_1 \rangle_\lambda = \lambda^2 (2X - 4M - 2C + H_l + H_r). \quad (115)$$

To get the complete v.e.v. of O_2 , we have to add (115) to (109). First, however, it is useful to notice that the square of the self-linking number

$$\text{slk}K = 2 \int_{\tilde{C}_2^0} \theta_{12} \quad (116)$$

can be written as

$$(\text{slk}K)^2 = 8(M + C - X); \quad (117)$$

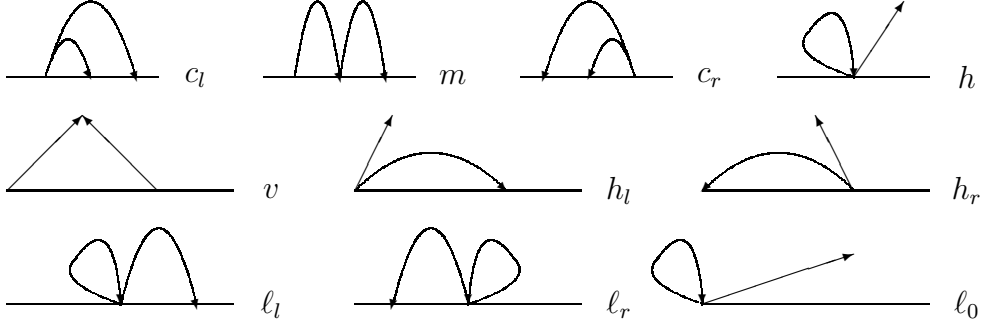


Figure 5: The principal-face diagrams

thus,

$$\langle O_2 \rangle_\lambda = -\frac{3}{8} \lambda^2 (\text{slk} K)^2 + \lambda^2 w_2, \quad (118)$$

with

$$w_2 = -X + V + H_l + H_r + I_0 - 2J_0. \quad (119)$$

We conclude this subsection by noticing that, if we take the unknot as a planar curve and choose its spanning surface to lie in the same surface, it is immediately proved that

$$w_2(\bigcirc) = 0. \quad (120)$$

Shortly we will prove that w_2 is a knot invariant; thus, (120) actually holds for any presentation of the unknot.

5.3.2 The invariance of the second-order term

Now we will show that the principal-face variation of w_2 vanishes; we will follow the approach of Ref. [10], which we have recalled in subsection 5.1.3.

Referring to figs. 3, 4 and 5, we start considering the principal-face con-

tributions

$$\begin{aligned}
\delta X &= c_l - m + c_r, \\
\delta V &= c_l + m + c_r - 2v + \ell_l + \ell_r, \\
\delta H_l &= -m + h - h_l + h_r - \ell_l, \\
\delta H_r &= -m - h - h_l + h_r - \ell_r, \\
\delta I_0 &= 2h_l + 2\ell_0, \\
\delta J_0 &= -v + h_r + \ell_0,
\end{aligned} \tag{121}$$

which by (119) imply

$$\delta w_2 = 0. \tag{122}$$

It should be clear from fig. 5 what c_l , m , c_r , v , h_l and h_r mean. To write the diagrams h , ℓ_l , ℓ_0 and ℓ_r , we need to introduce explicitly the map

$$\Phi : B^2 \longrightarrow \mathbb{R}^3 \tag{123}$$

that defines the surface Σ_K . The diagram h is given by

$$h = \int_{\tilde{C}_1^1} \theta_{12} \wedge \theta_{11}, \tag{124}$$

where θ_{11} is the pull back of the volume form ω through the map

$$\phi_{11}(\vec{x}) = \frac{\dot{\Phi}(x_1)}{|\dot{\Phi}(x_1)|}, \quad \vec{x} \in \tilde{C}_1^1. \tag{125}$$

and $\dot{\Phi}$ denotes the derivative of Φ in the direction tangent to the knot (notice that $\Phi(x_1)$ is on the knot). To describe the remaining diagrams, we have also to introduce Φ' , i.e., the derivative of Φ w.r.t. the other coordinate in the parametrization of the surface. In general the vector Φ' will *not* be orthogonal to $\dot{\Phi}$. To obtain an orthogonal vector we define

$$\Phi'_\perp = \Phi' - \frac{\Phi' \cdot \dot{\Phi}}{|\dot{\Phi}|^2} \dot{\Phi}. \tag{126}$$

Then the diagrams ℓ_l and ℓ_r read

$$\begin{aligned}
\ell_l &= \int_{\tilde{C}_2^0} [\theta_{12} \wedge \int_{\mathcal{U}_2} \theta_2], \\
\ell_r &= \int_{\tilde{C}_2^0} [\theta_{12} \wedge \int_{\mathcal{U}_1} \theta_1],
\end{aligned} \tag{127}$$

where the one-dimensional manifolds \mathcal{U}_i are defined as

$$\mathcal{U}_i = \{(u_1, u_2^1, u_2^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ / \quad [(u_1)^2 + (u_2^1)^2] |\dot{\Phi}(x_i)|^2 + (u_2^2)^2 |\Phi'_\perp(x_i)|^2 = 1, \\ u_1 + u_2^1 = 0\}, \quad (128)$$

and θ_i is the pull back of ω to \mathcal{U}_i through the map

$$\phi_i(u_1, u_2^1, u_2^2) = \frac{(u_2^1 - u_1) \dot{\Phi}(x_i) + u_2^2 \Phi'_\perp(x_i)}{|(u_2^1 - u_1) \dot{\Phi}(x_i) + u_2^2 \Phi'_\perp(x_i)|}. \quad (129)$$

Finally,

$$\ell_0 = \int_{\tilde{C}_0^1} [\theta_{01} \wedge \int_{\tilde{\mathcal{U}}_0} \tilde{\theta}_0], \quad (130)$$

where the one-dimensional manifold $\tilde{\mathcal{U}}_0$ is defined as

$$\tilde{\mathcal{U}}_0 = \{(u^1, u^2) \in \mathbb{R} \times \mathbb{R}^+ / (u^1)^2 |\dot{\Phi}(x_0)|^2 + (u^2)^2 |\Phi'_\perp(x_0)|^2 = 1\}, \quad (131)$$

and $\tilde{\theta}_0$ is the pull back of ω to $\tilde{\mathcal{U}}_0$ through the map

$$\tilde{\phi}_0(u_1, u_2) = \frac{u^1 \dot{\Phi}(x_0) + u^2 \Phi'_\perp(x_0)}{|u^1 \dot{\Phi}(x_0) + u^2 \Phi'_\perp(x_0)|}. \quad (132)$$

Notice that in (121) we have written only the non vanishing contributions. (Actually, more sophisticated arguments, s. App. C, show that also ℓ_l , ℓ_r and ℓ_0 vanish.) All other possible terms vanish for one of the following reasons:

1. we have to integrate a form on a space of lower dimension;
2. a factor θ_{ij}^2 appears, or
3. the push forward vanish because of a symmetry.

An example of the first case is the push forward of $[V]$ along the face obtained by sending 3 to 0 which gives

$$\int_{\tilde{\mathcal{U}}_0} [\int_{\tilde{C}_2^0} \theta_{10} \wedge \theta_{20}].$$

The second case happens, e.g., when we push forward $[V]$ along the face where we send 1 to 2. The third case occurs in the push forward of $[J_0]$ when

we send 1 to 2, the symmetry being the exchange of 1 with 2 which does not reverse the orientation of the manifold we are integrating over but changes the sign of the form to be integrated.

For the same reasons, the push forwards of $[X]$, $[I_0]$, $[J_0]$ and $[H_l] + [H_r]$ along the hidden faces vanish. The only non-trivial case is the push forward of $[V]$ along the hidden face where 1, 2 and 3 come together. This case is analyzed in App. C, and a vanishing theorem is proved.

These results together with (122) prove the following

Theorem 4 *The corrected second-order term w_2 is a topological invariant of the imbedding Σ_K of B^2 in \mathbb{R}^3 .*

As a consequence, if we deform the imbedding Σ_K by adding to it the boundary of a three-cycle, we can always move this deformation to infinity. Since all the vertices on Σ_K are connected through at most two θ 's to a point living on K , the deformation at infinity will not contribute. Therefore, w_2 actually depends only on K , and we have the following

Theorem 5 *The corrected second-order term w_2 is a knot invariant.*

The chord-diagram contribution $-X$ to w_2 is exactly the same that appears in the invariant studied in [21, 4]. This invariant is known to be equal to the second coefficient a_2 of the Alexander–Conway polynomial plus a constant term (viz., the value it takes on the unknot).

Notice that the chord diagram $-X$ alone is *not* a knot invariant. To get a knot invariant we have to add to it either the other terms that define w_2 —let call W their sum—or the diagram Y considered in [21, 4, 10] (viz., a diagram with a trivalent vertex in \mathbb{R}^3). Since both $-X + W$ and $-X + Y$ are knot invariants, also $T = W - Y$ is a knot invariant. Our claim is that T is trivial (i.e., it is the same for all knots). To prove it, it is enough to check that T takes the same value on two knots K_+ and K_- that differ only around a chosen crossing. We notice that the difference $T(K_+) - T(K_-)$ comes from a singularity at the crossing point where the flip occurs—as in [4]—or along the line where the two spanning surfaces get to intersect. However, it is not difficult to check that such singularities do not arise; so T is a constant.

Therefore, w_2 is equal to a_2 plus a constant. However, since by definition $a_2(\bigcirc) = 0$, (120) implies

$$w_2 = a_2, \tag{133}$$

and Conjecture 3 on page 19 is satisfied at this order.

As a concluding remark, we notice that in passing from Y to W one of the integrations on the knot is replaced by an integration on the spanning surface; so it should be possible to relate W and Y directly via Stokes's theorem.

5.3.3 Higher orders

Thm. 3 ensures that a quantum observable \mathcal{O} extending \mathcal{O}_0 exists. Its v.e.v. at order λ^n will be given by diagrams containing n propagators connecting points on the knot and/or on the spanning surface. Of course, the restrictions given in subsection 5.2.1 for the v.e.v. of \mathcal{O}_0 do not hold anymore; in particular, the vertices on the knot will not necessarily be univalent and the vertices on the spanning surface will not necessarily be bivalent (we have already seen a counterexample at the second order). However, no loops on the surface will appear (since the corrections must vanish when the spanning surface is boundaryless). Moreover, since the v.e.v. of \mathcal{O}_0 vanishes at odd order, we do not need odd-order corrections.

The combinatorics of these diagrams will be dictated by the specific form of the corrected observable \mathcal{O} . What we expect, by field-theoretical arguments, is that these combinations of diagrams will be metric independent, i.e., will be the sum of invariants possibly times powers of the self-linking number, i.e., “isolated chords” in the diagrams. The true invariants will then be obtained by factorizing the isolated chords.

A rigorous mathematical proof that they are actually knot invariants will simply require checking that the principal-face contributions of the diagrams that sum up cancel each other, for Thm. 7 on page 44 ensures that the push forwards along hidden faces always vanish.

Notice that now we could also throw away the BF field theory and directly study δ -closed combinations of diagrams with vertices on the knot and/or on the spanning surface. By Thm. 7, these will yield knot invariants as well as higher-degree cohomology classes on the space of imbeddings (the degree being given by $2l - n - 2t$ where l is the number of propagators, n the number of points on the knot and t the number of points on the surface).

6 A glimpse to higher dimensions

There is no problem in defining the Abelian BF theory in any dimension: just take A and B to be fields taking values in $\Omega^p(M)$ and $\Omega^q(M)$ respectively, with $p + q + 1 = d$ and $d = \dim M$. The classical action (19) can easily be extended to a BV action. The partition function then is known to be equal to the Ray–Singer torsion or to its inverse (depending on p) [29, 8]. Moreover, it is not difficult to generalize the observable (53), where now $\Sigma \in H_{d-1}(M, \partial M)$. The classical part of this observable reads

$$\gamma_{\text{cl}}^\Sigma = \int_\Sigma B \wedge A, \quad (134)$$

and satisfies

$$s\gamma_{\text{cl}}^\Sigma = 0, \quad \text{on shell}, \quad (135)$$

where *on shell* means modulo the classical equations of motion,

$$dA = 0, \quad dB = 0, \quad (136)$$

and s is the BRST operator

$$sA = dc, \quad sB = d\psi \quad (137)$$

(now c and ψ are a $(p-1)$ - and a $(q-1)$ -form respectively).

If Σ_K is a spanning surface for a $(d-2)$ -knot K (i.e., an imbedding of S^{d-2} in \mathbb{R}^d), then

$$s\gamma_{\text{cl}}^{\Sigma_K} = \oint_K [\psi \wedge A + (-1)^q B \wedge c], \quad \text{on shell}. \quad (138)$$

To get an on-shell s -closed functional, we have to add to γ^{Σ_K} another term canceling the r.h.s. of (138). We first notice that $\gamma^{(K,x_0)}$ as in (61) can be generalized in any dimension, where now η is the tautological $(d-3)$ -form on the configuration space of $K \setminus x_0$. An explicit computation shows that

$$s\gamma_{\text{cl}}^{(K,x_0)} = (-1)^{d+1+p} \oint_K [\psi \wedge A + (-1)^{q+d+1} B \wedge c], \quad \text{on shell}. \quad (139)$$

Therefore, *in odd dimension* we can define the following on-shell s -closed functional

$$\gamma_{\text{cl}}^{(\Sigma_K, K, x_0)} = \gamma_{\text{cl}}^{\Sigma_K} + (-1)^{p+1} \gamma_{\text{cl}}^{(K, x_0)}. \quad (140)$$

Then, starting from (140), the BV procedure will yield a σ -closed observable.

Finally, we would like to consider an object like \mathcal{O}_0 in (66), for its v.e.v. should be related to the Alexander–Conway polynomial (or its inverse, depending on p). Of course, we do not expect \mathcal{O}_0 to be an observable, so we should look for corrections as explained in subsection 4.4.

Notice that in any dimension it is possible to define linear combinations \mathcal{A} and \mathcal{B} generalizing (26), (27) and (28) (and including the whole set of ghosts for ghosts).

Moreover, in odd dimension the classical action is invariant under $(A, B) \rightarrow (B, A)$ while $\gamma_{\text{cl}}^{(\Sigma_K, K, x_0)}$ is odd under it. Thus, their BV extensions will share the same property under $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{B}, \mathcal{A})$. This leads to proving a generalization of Thm. 3 on page 18 stating that \mathcal{O}_0 is never anomalous. We have only to check that the form degree of the one-ghost-number component of any form with well-defined parity under the above transformation never matches with the dimension of a nontrivial homology space of $C_n(\mathbb{R}^d)$. As a matter of fact, these dimensions are multiples of $(d - 1)$. On the other hand, forms with well-defined parity are obtained by products of $\mathcal{B} \wedge \mathcal{A}$ and $\tilde{\mathcal{B}} \wedge \mathcal{A}$, both of which are overall $(d - 1)$ -forms, times a certain number r of tautological $(d - 3)$ -forms η ; thus, the form degree of the one-ghost-number component will be congruent to $-1 - 2r \bmod (d - 1)$. Then our claim follows from the fact that

$$2r + 1 \equiv 0 \bmod (d - 1)$$

has no solutions if d is odd.

The v.e.v. of \mathcal{O} should then yield metric-independent functionals of the knot and its spanning surface. Eventually, if a vanishing theorem holds, these functionals will be knot invariants. Hence, we could compute numerical knot invariants (presumably the coefficients of the Alexander–Conway polynomial or its inverse) in any odd dimension in terms of integrals over the configuration spaces of points on the knot and on its spanning surface.

Via Stokes’s theorem, the second-order invariants should correspond, up to a constant term, to the ones proposed in [9].

As a final remark, we notice that in three dimensions we could have chosen A to be a zero-form and B a two-form. In this case, the v.e.v. of \mathcal{O} should give directly the Alexander–Conway polynomial instead of its inverse.

7 Conclusions

In this paper we have considered a new way of obtaining knot invariants from a TQFT.

The nice feature of our theory is that it is Abelian. What makes things non-trivial is a rather involved observable, which can be defined only in the context of BV formalism; yet, as observed in the last section, it can be generalized in any odd dimension.

In the three-dimensional case, we have shown that at the second order the theory actually produces a numerical knot invariant which, despite the fact it is not new, comes out written in an entirely new way.

The next task is to find the other corrections to the observable and to evaluate higher-order v.e.v.'s, in three dimensions as well as in any odd dimension.

Of course, an alternative way would be working directly on the space of surface-plus-knot diagrams, as described in subsection 5.1.3, and try to find combinations whose differential vanishes. This would allow studying higher-degree forms on the space of imbeddings as well.

Notice that while the Chern-Simons and the BF theory with a cosmological term produce the whole set of HOMFLY polynomials and their “colored” generalizations, pure BF theories, both Abelian and non-Abelian, give only the Alexander–Conway polynomial.

However, it is possible that even more involved observables exist whose v.e.v. is a more general knot invariant. As a matter of fact, pure BF theory comes out naturally as a particular limit of the v.e.v. of a cabled Wilson loop in the theory with a cosmological term [12]. This limit corresponds to the first diagonal in the (h, d) expansion of the colored Jones function [25]. A generalization of the computation done in [12] should give the observables whose v.e.v.'s correspond to the upper diagonals in this expansion. Then a careful study of the “Abelianizing limit” described in the Introduction should yield the corresponding observables for the Abelian theory.

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A The treatment of the harmonic zero modes

If the ω -Laplacian $d_\omega^* d_\omega + d_\omega d_\omega^*$ has zero modes, the formulae in Sec. 3 have to be slightly modified. We will essentially follow the approach explained in [8], adapting it to the BV formalism. Notice that we suppose here that $H^1(M, d_\omega)$ is not trivial, but we go on assuming that $H^0(M, d_\omega)$ is trivial since we want the symmetry group to act freely.

The first step is to modify the BV action (23) so as to include the symmetry obtained by adding an ω - $[(-\omega)]$ -harmonic form to A $[B]$; viz.,

$$S^\omega \longrightarrow S^\omega + \sum_{\alpha=1}^{b_1[\omega]} \left[\int_M (A^* \wedge k^\alpha \varphi_\alpha + B^* \wedge p^\alpha \bar{\varphi}_\alpha) + \bar{k}_\alpha^\dagger \chi^\alpha + \bar{p}_\alpha^\dagger \pi^\alpha \right], \quad (141)$$

where

- $b_1[\omega] = \dim H^1(M, d_\omega)$;
- $\{\varphi_\alpha\}$ $[\{\bar{\varphi}_\alpha\}]$ is an orthonormal basis of ω - $[(-\omega)]$ -harmonic one-forms; it is convenient to choose the normalization

$$\langle \varphi_\alpha, \varphi_\beta \rangle = \langle \bar{\varphi}_\alpha, \bar{\varphi}_\beta \rangle = v \delta_{\alpha\beta}, \quad (142)$$

where v is the volume of the manifold M (which we suppose to be compact);

- k^α and p^α are constant fields with ghost number one;
- \bar{k}^α and \bar{p}^α are constant fields with ghost number minus one, and
- χ^α and π^α are constant fields with ghost number zero.

Notice that now the action (141) is not metric independent; as a matter of fact, the choice of the bases $\{\varphi_\alpha\}$ and $\{\bar{\varphi}_\alpha\}$ requires a volume form. Thus, we cannot expect the partition function to be a topological invariant. However, it is not difficult to see that the argument given in Sec. 3 to prove that the partition function depends only on the cohomology class of ω still holds.

The action of the σ^ω operator is the same as in (24) and (25) but on A and B where it acts as follows:

$$\sigma^\omega A = d_\omega c + \sum_{\alpha=1}^{b_1[\omega]} k^\alpha \varphi_\alpha, \quad \sigma^\omega B = \bar{d}_\omega \psi + \sum_{\alpha=1}^{b_1[\omega]} p^\alpha \bar{\varphi}_\alpha. \quad (143)$$

Moreover, we have

$$\begin{aligned} \sigma^\omega k_\alpha^\dagger &= \int_M A^* \wedge \varphi_\alpha, \quad \sigma^\omega k_\alpha = 0, \\ \sigma^\omega p_\alpha^\dagger &= \int_M B^* \wedge \bar{\varphi}_\alpha, \quad \sigma^\omega p_\alpha = 0, \end{aligned} \quad (144)$$

$$\begin{aligned} \sigma^\omega \chi_\alpha^\dagger &= -\bar{k}_\alpha^\dagger, \quad \sigma^\omega \bar{k}_\alpha^\dagger = 0, \quad \sigma^\omega \bar{k}^\alpha = \chi^\alpha, \quad \sigma^\omega \chi^\alpha = 0, \\ \sigma^\omega \pi_\alpha^\dagger &= -\bar{p}_\alpha^\dagger, \quad \sigma^\omega \bar{p}_\alpha^\dagger = 0, \quad \sigma^\omega \bar{p}^\alpha = \pi^\alpha, \quad \sigma^\omega \pi^\alpha = 0. \end{aligned} \quad (145)$$

The gauge-fixing fermion defined in (40) has now to be modified as

$$\Psi \longrightarrow \Psi + \sum_{\alpha=1}^{b_1[\omega]} \left(\bar{k}^\alpha \langle \varphi_\alpha, A \rangle + \bar{p}^\alpha \langle \bar{\varphi}_\alpha, B \rangle \right) \quad (146)$$

in order to fix the new symmetries (143). With this choice of gauge, we have

$$Z[M, \omega] = \int [D\Phi] \prod_{\alpha=1}^{b_1[\omega]} dk^\alpha d\bar{k}^\alpha dp^\alpha d\bar{p}^\alpha d\chi^\alpha d\pi^\alpha \exp(iS_{\text{g.f.}}^\omega), \quad (147)$$

where

$$[D\Phi] = [DA DB Dc D\bar{c} D\psi D\bar{\psi} Dh_c Dh_\psi], \quad (148)$$

and $S_{\text{g.f.}}^\omega$ is the following modification of (42)

$$S_{\text{g.f.}}^\omega \longrightarrow S_{\text{g.f.}}^\omega + \sum_{\alpha=1}^{b_1[\omega]} \left[v (\bar{k}^\alpha k^\alpha + \bar{p}^\alpha p^\alpha) + \chi^\alpha \langle \varphi_\alpha, A \rangle + \pi^\alpha \langle \bar{\varphi}_\alpha, B \rangle \right] \quad (149)$$

An explicit computation of (147), in zeta-function regularization, shows that

$$Z[M, \omega] = v^{2b_1[\omega]} \mathcal{T}(M, d_\omega); \quad (150)$$

thus, apart from a volume factor, the partition function is still a topological invariant.

B The first correction to the observable \mathcal{O}_0

In this appendix we solve eqn. (74) for $n = 1$. By (67), we have

$$\sigma\mathcal{O}_1 = \frac{1}{2} \mathcal{O}_0 \left(\gamma^{(\Sigma_K, K, x_0)}, \gamma^{(\Sigma_K, K, x_0)} \right); \quad (151)$$

thus, by (77),

$$\sigma U_1 = \frac{1}{2} \left(\gamma^{(\Sigma_K, K, x_0)}, \gamma^{(\Sigma_K, K, x_0)} \right). \quad (152)$$

An explicit computation shows that the nonvanishing terms of (152) are given by

$$2\sigma U_1 = \left(\int_{C_2(K \setminus x_0)} A_1 \eta_{12} B_2, \int_{\Sigma_K} [B^*(\psi - \psi(x_0)) + (c - c(x_0))A^*] \right). \quad (153)$$

(We suppress all the \wedge symbols for simplicity.)

Since the antibracket contracts fields evaluated at the same point, we have the following identities

$$\begin{aligned} \left(\int_{C_2(K \setminus x_0)} A_1 \eta_{12} B_2, \int_{\Sigma_K} B^*(\psi - \psi(x_0)) \right) &= \left(\int_{C_2(K \setminus x_0)} A_1 \eta_{12} B_2 (\psi_2 - \psi_0), \int_{\Sigma_K} B^* \right), \\ \left(\int_{C_2(K \setminus x_0)} A_1 \eta_{12} B_2, \int_{\Sigma_K} (c - c(x_0))A^* \right) &= - \left(\int_{C_2(K \setminus x_0)} (c_1 - c_0) A_1 \eta_{12} B_2, \int_{\Sigma_K} A^* \right); \end{aligned} \quad (154)$$

moreover, since the antibracket of A with A^* is the same as the antibracket of B with B^* , we have also the following identities

$$\begin{aligned} \left(\int A_1 \eta_{12} B_2 \psi_2, \int B^* \right) &= \left(\int A_1 \eta_{12} A_2 \psi_2, \int A^* \right) - \left(\int B_1 \eta_{12} A_2 \psi_2, \int B^* \right), \\ \left(\int A_1 c_1 \eta_{12} B_2, \int A^* \right) &= \left(\int B_1 c_1 \eta_{12} B_2, \int B^* \right) - \left(\int B_1 c_1 \eta_{12} A_2, \int A^* \right). \end{aligned} \quad (155)$$

By (154) and (155), (153) now reads

$$\begin{aligned} 2\sigma U_1 &= \left(\int_{C_2(K \setminus x_0)} [A_1 \eta_{12} A_2 \psi_2 + B_1 c_1 \eta_{12} A_2 + c_0 A_1 \eta_{12} B_2], \int_{\Sigma_K} A^* \right) + \\ &- \left(\int_{C_2(K \setminus x_0)} [B_1 \eta_{12} A_2 \psi_2 + B_1 c_1 \eta_{12} B_2 + A_1 \eta_{12} B_2 \psi_0], \int_{\Sigma_K} B^* \right). \end{aligned} \quad (156)$$

If now we define γ_{ABB} and γ_{BAA} as in (80) and (81), we see that

$$\begin{aligned} \frac{1}{2} \left(\gamma^{(\Sigma_K, K, x_0)}, \gamma^{(\Sigma_K, K, x_0)} \right) &= - \left(\sigma \gamma_{ABB}, \int_{\Sigma_K} B^* \right) - \left(\sigma \gamma_{BAA}, \int_{\Sigma_K} A^* \right) = \\ &= \Delta \left[(\sigma \gamma_{ABB}) \int_{\Sigma_K} B^* + (\sigma \gamma_{BAA}) \int_{\Sigma_K} A^* \right]. \end{aligned} \quad (157)$$

Since

$$\Delta \left[\gamma_{ABB} (\sigma \int_{\Sigma_K} B^*) + \gamma_{BAA} (\sigma \int_{\Sigma_K} A^*) \right] = 0,$$

we get

$$\frac{1}{2} \left(\gamma^{(\Sigma_K, K, x_0)}, \gamma^{(\Sigma_K, K, x_0)} \right) = -\Delta \sigma u_1 = \sigma \Delta u_1, \quad (158)$$

where we have used (11), and u_1 is defined in (79). By (152) and (77), we arrive at (78) up to a σ -closed term. In particular, the Δ -, σ -closed correction leading to (85) can be written as

$$U_1 \rightarrow U_1 - \frac{k}{2} \delta U_1 + l\tau + i\lambda m, \quad (159)$$

with τ defined in (84), and

$$\delta U_1 = \Delta \delta u_1, \quad (160)$$

where

$$\delta u_1 = \gamma_A^K (\gamma_B^K)^2 \int_{\Sigma_K} B^* + \gamma_B^K (\gamma_A^K)^2 \int_{\Sigma_K} A^*. \quad (161)$$

In fact,

$$i\lambda \Delta \delta u_1 = \sigma \delta u_1 - \Omega \delta u_1 = -2\tau^2 - \Omega \delta u_1; \quad (162)$$

therefore, by Statement 1 on page 7,

$$\langle i\lambda \Delta \delta u_1 \rangle_\lambda = -2 \langle \tau [K]^2 \rangle_\lambda. \quad (163)$$

C Vanishing theorems

C.1 The hidden-face contribution to dV

In this subsection we show that dV does not have hidden-face contributions. The only nontrivial hidden face is the one where the three point come together (it is easy to show that the pushforward of $[V]$ along all other hidden faces vanishes). We will show that the push forward of $[V]$ vanishes on this face, too.

To begin with, we parametrize the three points on this face, which we will denote by S , as

$$\begin{aligned} x_1 &= \Phi(y) + r \frac{\dot{\Phi}(y)}{|\dot{\Phi}(y)|} u_1 + \cdots, \\ x_2 &= \Phi(y) + r \frac{\dot{\Phi}(y)}{|\dot{\Phi}(y)|} u_2 + \cdots, \\ x_3 &= \Phi(y) + r \left(\frac{\dot{\Phi}(y)}{|\dot{\Phi}(y)|} u_3 + \frac{\Phi'_\perp(y)}{|\Phi'_\perp(y)|} u_4 \right) + \cdots, \end{aligned} \quad (164)$$

where $y \in S^1$, $r \rightarrow 0^+$ and the u variables live in the two-dimensional manifold

$$\mathcal{U} = \{ (u_1, u_2, u_3, u_4) \in \mathbb{R}^3 \times \mathbb{R}^+ / u_1 < u_2, \sum_{i=1}^4 (u_i)^2 = 1, \sum_{i=1}^3 u_i = 0, (u_i, 0) \neq (u_3, u_4), i = 1, 2 \}. \quad (165)$$

Following the approach of [10], we want to represent $[V]$ on S as the pull back of a universal form $\hat{\lambda}$. More precisely, we introduce the maps

$$\begin{aligned} \psi : \tilde{C}_1^0 &\rightarrow S^2, \\ y &\mapsto \frac{\dot{\Phi}(y)}{|\dot{\Phi}(y)|} \end{aligned} \quad (166)$$

$$\begin{aligned} \chi : \tilde{C}_1^0 &\rightarrow S^2, \\ y &\mapsto \frac{\Phi'_\perp(y)}{|\Phi'_\perp(y)|} \end{aligned} \quad (167)$$

and

$$f = (\psi, \chi) : \tilde{C}_1^0 \rightarrow S^2 \times S^2. \quad (168)$$

Then we consider the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\hat{f}} & \mathcal{U} \times S^2 \times S^2 \\ \downarrow \pi & & \downarrow \hat{\pi} \\ \tilde{C}_1^0 & \xrightarrow{f} & S^2 \times S^2 \end{array}, \quad (169)$$

where \hat{f} maps S into the parametrization \mathcal{U} of the blow up times the value of f , while π and $\hat{\pi}$ are projections along the fibers. Eventually, we notice that there exists a universal four-form $\hat{\lambda}$ on $\mathcal{U} \times S^2 \times S^2$ such that

$$[V] = \hat{f}^* \hat{\lambda} \quad \text{and} \quad \pi_*[V] = f^* \hat{\pi}_* \hat{\lambda}. \quad (170)$$

To write $\hat{\lambda}$ explicitly, we have to introduce the maps ϕ_{13} , ϕ_{23} and ϕ defined as

$$\begin{aligned} \phi_{i3} : \mathcal{U} \times S^2 \times S^2 &\rightarrow S^2, \quad i = 1, 2, \\ (u, a, b) &\mapsto \frac{(u_3 - u_i)a - u_4 b}{|(u_3 - u_i)a - u_4 b|} \end{aligned} \quad (171)$$

and

$$\phi = (\phi_{13}, \phi_{23}) : \mathcal{U} \times S^2 \times S^2 \rightarrow S^2 \times S^2. \quad (172)$$

Then we have

$$\hat{\lambda} = \phi^*(\omega_1 \wedge \omega_2), \quad (173)$$

where ω_1 and ω_2 are the $SO(3)$ -invariant unit elements on the two spheres.

Now we notice that ϕ and $\hat{\pi}$ are equivariant maps under the action of $SO(3)$ (we mean the diagonal action on $S^2 \times S^2$ and the trivial action on \mathcal{U}), and that $\omega_1 \wedge \omega_2$ is $SO(3)$ -invariant. Thus, $\hat{\pi}_*\hat{\lambda}$ is rotationally invariant as well. Since it is a two-form and the only rotational invariant forms (up to multiplication by a scalar) on S^2 are 1 and ω , we conclude that $\hat{\pi}_*\hat{\lambda}$ reads

$$\hat{\pi}_*\hat{\lambda} = c_1\omega_1 + c_2\omega_2, \quad (174)$$

where c_1 and c_2 are constants.

Finally, we consider the (orientation reversing) automorphism Θ of S^2 that maps a point into its antipode, and notice that $\Theta^*\omega = -\omega$. Therefore, if we consider the diagonal extension of Θ to $S^2 \times S^2$, we see that

$$\Theta^*\hat{\pi}_*\hat{\lambda} = -\hat{\pi}_*\hat{\lambda}. \quad (175)$$

On the other hand, ϕ and $\hat{\pi}$ are equivariant maps under the action of Θ (we mean the diagonal action on $S^2 \times S^2$ and the trivial action on \mathcal{U}). Since $\Theta^*(\omega_1 \wedge \omega_2) = \omega_1 \wedge \omega_2$, we conclude that

$$\Theta^*\hat{\pi}_*\hat{\lambda} = \hat{\pi}_*\hat{\lambda}. \quad (176)$$

This, together with (175), implies that $\hat{\pi}_*\hat{\lambda}$ vanishes. Thus, owing to (170), we have proved the following

Theorem 6 *The push forward of $[V]$ along the hidden face where all the points collapse together vanishes.*

C.2 The general vanishing theorem

A generalization of the previous argument leads to proving a more general vanishing theorem for diagrams involving tautological forms connecting points on the knot and/or on its spanning surface.

The first step is to generalize the commutative diagram (169). In general, the stratum S we are considering will have a natural projection π to a configuration space with less points [which will replace \tilde{C}_1^0 in the lower left corner of (169)]. Suppose we start with n points on the knot and t points on the surface, i.e., with the $(n + 2t)$ -dimensional configuration space \tilde{C}_n^t . We want q points on the knot and s points on the surface to collapse. There are three cases:

Case 1) $q = 0$ and the s points collapse together on the surface;

Case 2) the points collapse on the base point;

Case 3) otherwise.

In case 1, π projects S to \tilde{C}_n^{t-s+1} ; in case 2, to \tilde{C}_{n-q}^{t-s} ; in case 3, to \tilde{C}_{n-q+1}^{t-s} . The dimension of the fiber, which we shall denote by D , can be computed noticing that $\dim S = \dim \tilde{C}_n^t - 1$. In case 1, we have $D = 2s - 3$; in case 2, $D = q + 2s - 1$; in case 3, $D = q + 2s - 2$.

We will prove that the push forward along a face of \tilde{C}_n^t vanishes if $D > 0$. Since $D = 0$ is satisfied only in case 2, with $(q = 1, s = 0)$, or in case 3, with either $(q = 0, s = 1)$ or $(q = 2, s = 0)$, we have the following

Theorem 7 *The push forward along a hidden face always vanishes; so does the push forward along a principal face unless it is simple.*

(For a definition of principal, hidden and simple principal faces, s. page 24.)

In particular, this proves that the contributions ℓ_l , ℓ_r and ℓ_0 —considered in subsection 5.3.2—vanish.

Proof First we split the form on S into the product $\lambda_1 \wedge \pi^* \lambda_2$. The propagators that define λ_2 connect either two points that do not collapse, or a point that does not collapse with a point that does, or two of the q points on the knot that collapse together. The propagators that define λ_1 connect two collapsing points at least one of which is on the surface.

It follows that λ_1 is written in terms of products of pull backs of ω via maps that are linear combinations of the unit vectors $\dot{\Phi}/|\dot{\Phi}|$ and $\Phi'_\perp/|\Phi'_\perp|$. Thus, S will be mapped to a proper submanifold of $(S^2)^n$ unless $n = 0$, $n = 1$ or $n = 2$. This means that λ_1 vanishes unless it is a zero-, a two- or a four-form. In the first case, however, $\pi_*\lambda_1$ clearly vanishes unless $D = 0$.

Now the idea is to generalize the commutative diagram (169) and write λ_1 in terms of a universal form $\hat{\lambda}_1$:

$$\lambda_1 = \hat{f}^*\hat{\lambda}_1 \quad \text{and} \quad \pi_*\lambda_1 = f^*\hat{\pi}_*\hat{\lambda}_1. \quad (177)$$

The left column of (169) is unchanged if we still denote by \mathcal{U} the parametrization of the blow up. Explicitly the manifold \mathcal{U} reads

$$\mathcal{U} = \{ ((u_1^1, u_1^2), \dots, (u_s^1, u_s^2)) \in (\mathbb{R} \times \mathbb{R}^+)^s / \sum_{i=1}^s \sum_{\alpha=1}^2 (u_i^\alpha)^2 = 1, \sum_{i=1}^s u_i^\alpha = 0, u_i^\alpha \neq u_j^\alpha \forall \alpha \text{ if } i \text{ and } j \text{ are connected} \}, \quad (178)$$

in case 1;

$$\mathcal{U} = \{ (u_1, \dots, u_q, (u_{q+1}^1, u_{q+1}^2), \dots, (u_{q+s}^1, u_{q+s}^2)) \in \mathbb{R}^q \times (\mathbb{R} \times \mathbb{R}^+)^s / u_1 < \dots < u_q, \sum_{i=1}^q (u_i)^2 + \sum_{i=q+1}^{q+s} \sum_{\alpha=1}^2 (u_i^\alpha)^2 = 1, u_i^\alpha \neq u_j^\alpha \forall \alpha \text{ and } (u_i, 0) \neq (u_j^1, u_j^2) \text{ if } i \text{ and } j \text{ are connected} \}, \quad (179)$$

in case 2, and

$$\mathcal{U} = \{ (u_1, \dots, u_q, (u_{q+1}^1, u_{q+1}^2), \dots, (u_{q+s}^1, u_{q+s}^2)) \in \mathbb{R}^q \times (\mathbb{R} \times \mathbb{R}^+)^s / u_1 < \dots < u_q, \sum_{i=1}^q (u_i)^2 + \sum_{i=q+1}^{q+s} \sum_{\alpha=1}^2 (u_i^\alpha)^2 = 1, \sum_{i=1}^q u_i + \sum_{i=q+1}^{q+s} u_i^1 = 0, u_i^\alpha \neq u_j^\alpha \forall \alpha \text{ and } (u_i, 0) \neq (u_j^1, u_j^2) \text{ if } i \text{ and } j \text{ are connected} \}, \quad (180)$$

in case 3.

The universal form $\hat{\lambda}_1$ is written in terms of pullbacks of ω via $\hat{\phi}$ -maps defined as follows:

$$\hat{\phi}_{ij} = \frac{(u_j^1 - u_i)a + u_j^2 b}{|(u_j^1 - u_i)a + u_j^2 b|}, \quad \text{if } i \leq q \text{ and } q < j \leq q + s, \quad (181)$$

$$\hat{\phi}_{ij} = \frac{(u_j^1 - u_i^1)a + (u_j^2 - u_i^2)b}{|(u_j^1 - u_i^1)a + (u_j^2 - u_i^2)b|}, \quad \text{if } q < i < j \leq q + s, \quad (182)$$

and $\hat{\phi}_{ji} = -\hat{\phi}_{ij}$.

We must now distinguish two subcases: viz., when $\hat{\lambda}_1$ is a two-form and when it is a four-form.

In the first subcase, $\hat{\lambda}_1$ is obtained via pull back through a single $\hat{\phi}$ -map to S^2 . Rotational invariance shows that $\hat{\pi}_*\hat{\lambda}_1$ does not vanish only if it is a zero- or a two-form; in the former instance it is a constant, in the latter it is linear combination of ω_1 and ω_2 . However, $\hat{\pi}_*\hat{\lambda}_1$ must be odd under the action of the automorphism Θ that maps a point on S^2 into its antipode since ω is odd. Therefore, it does not vanish only in the latter instance. However, if both $\hat{\lambda}_1$ and $\hat{\pi}_*\hat{\lambda}_1$ are two-forms, we have $D = 0$.

In the second subcase, $\hat{\lambda}_1$ is obtained via pull back through a product of two $\hat{\phi}$ -maps to $S^2 \times S^2$, as, e.g., in (172). Rotational invariance shows that $\hat{\pi}_*\hat{\lambda}_1$ does not vanish only if it is a zero-, a two- or a four-form; in the first instance it is a constant, in the second it is a linear combination of ω_1 and ω_2 , in the third it is a multiple of $\omega_1 \wedge \omega_2$. However, $\hat{\pi}_*\hat{\lambda}_1$ must be even under the diagonal action of the automorphism Θ since $\omega_1 \wedge \omega_2$ is even. Thus, $\hat{\pi}_*\hat{\lambda}_1$ does not vanish only if it is a zero- or a four-form. Since $\hat{\lambda}_1$ is a four form, we have $D = 4$ in the former instance and $D = 0$ in the latter.

Therefore, to complete the proof we have only to show that, in the former instance, $\hat{\pi}_*\hat{\lambda}_1$ vanishes. This happens since, if $D = 4$, $\hat{\pi}_*$ selects the $(4, 0)$ -component of the four-form $\hat{\lambda}_1$ on $\mathcal{U} \times (S^2 \times S^2)$. This component, however, vanishes since, for fixed $(a, b) \in S^2 \times S^2$, $\hat{\phi}$ maps the four-dimensional manifold \mathcal{U} into a two-dimensional submanifold of $S^2 \times S^2$ [this submanifold is parametrized by two unit vectors $z_i(u; a, b)$ satisfying $z_i \cdot (a \times b) = 0$, $i = 1, 2$].

This concludes our proof.

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